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Solidarity Value for Network Games

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Abstract: The solidarity value is an allocation rule for cooperative games that redistributes payoffs by averaging marginal contributions, promoting mutual support among coalition members. In this paper, we extend the solidarity principle to introduce an allocation rule for network games, where feasible coalitions are constrained by a communication graph. The key idea is that, within any connected coalition, a player whose marginal contribution exceeds the average marginal contribution of that coalition offers a portion of their surplus to support weaker members whose marginal contribution fall below the average. This redistribution reflects a solidarity-based adjustment within the graph-connected coalitions. We provide an axiomatic characterization of the solidarity value for networks games along the lines of Myerson value.

Keywords: Cooperative games, Cooperation graph, Myerson value, Network games, Solidarity value

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1. Introduction

Cooperative game theory examines how a group of players can collaborate and fairly divide the collective benefits generated by their cooperation. Each possible subset of players, known as a coalition, is associated with a specific value reflecting its potential gains. A central focus is the distribution of the total value generated by the grand coalition that is, when all players cooperate using well-defined allocation rules. Among the most widely studied rules the Shapley value [4] offering principled ways to assign payoffs based on players' individual contributions. The solidarity value [2] is a new value function which reflects some social behaviour of players in coalitions arising out of solidarity considerations. The modelling of cooperative behaviour among agents embedded in a network has become increasingly important in economics, political science, social science and technological domains. The foundational work in this direction was introduced by Myerson (1977) through the concept of communication situations. Here, the players are connected via an undirected graph representing who can communicate or cooperate directly. A coalition is considered feasible only if its members form a connected sub-graph i.e., there exists a communication path between every pair of players in the group. To account for these restrictions, Myerson proposed modifying the original game to a restricted game, and then applying the Shapley value to it. The resulting payoff distribution is known as the Myerson value, which reflects both players' contributions and the structural limitations imposed by the communication network. The main contribution of this paper is to extend the solidarity value to network situations.

1.1 Preliminieries

1.1.1 Solidarity Value for Cooperative Games

Let the player set be given. Let 2^N denote the set of all coalitions obtained from the player set A TU cooperative game or simply a cooperative game is the pair (N,v) where N={1,2,…,n} is a set of n players, called the grand coalition and v is the function v:2^N→R such that v(∅)=0. If no ambiguity about the player set N arises, we denote a TU game by its characteristic function v only. Let G_0 (N) denote the class of all TU games with player set N. Recall that a solution of a TU-game with n players is an n-dimensional vector representing a distribution of payoffs. A value function is an allocation rule on a subset C of G_0 (N) is a function that assigns a solution to any game in C.

Definition 1. The Shapley value φ_i^S of player i with respect to a game $v \in G_0(N)$ is a weighted average value of the marginal contribution $v(S) - v(S\backslash\{i\})$ of player i alone in all combinations, which is defined by

$$\varphi_i^s(N,v) = \sum_{\{S:i\in S\in P(N)\}} \frac{(|S| - 1)!\,(|N| - |S|)!}{|N|!}[v(S) - v(S\backslash\{i\})].$$

Definition 2. Let $T \in 2^N$ and $v \in G_0(N)$ the quantity

$$A^v(T) = \frac{1}{|T|}\sum_{k\in T}[v(T) - v(T \setminus k)]$$

is called the average marginal contribution of a player of the coalition T.

Definition 3. Given a game $v \in G_0(N)$ Player $i \in N$ is called a A-null player if $A^v(T) = 0$ for every coalition $T \subseteq N$ containing i.

Definition 4. A function $\varphi_i^{sol}: G_0(N) \to (\mathbb{R}_+^n)^{2^N}$ is said to be a solidarity value function on $G_0(N)$ if it satisfies the following four axioms.

Axiom S1. (Efficiency) If $v \in G_0(N)$

$$\sum_{i\in W} \varphi_i^{sol}(N,v) = v(N)$$

Axiom S2. (A-null player) If $v \in G_0(N)$ **and** $i \in T \in 2^N$ is a A-null player, that is $A^v(T) = 0$ then

$$\varphi_i^{sol}(N,v) = 0 \quad \forall\ i \in T \subset N,$$

Axiom S3.(Symmetry) If $v \in G_0(N)$ and $i,j \in N$ are symmetric i.e., $v(S \cup i) = v(S \cup j)$ holds for any $S \in 2^{N\backslash\{i,j\}}$, then

$$\varphi_i^{sol}(N,v) = \mu_j^{sol}(N,v),$$

Axiom S4.(Additivity) For $v_1, v_2 \in G_0(N)$, define $v_1 + v_2 \in G_0(N)$ by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for each $S \in 2^N$. If $v_1, v_2 \in G_0(N)$ then

$$\varphi_i^{sol}(N, v_1 + v_2) = \varphi_i^{sol}(N, v_1) + \varphi_i^{sol}(N, v_2)$$

Theorem 1. *For* $\emptyset \neq T \in 2^N$*, the game* u_T *i.e.,*

$$u_T(S) = \begin{cases} \binom{|S|}{|T|}^{-1}, & if\ S \supseteq T \\ 0, & otherwise \end{cases}$$

has the following properties:

$$(i)\, u_T(T) = 1, \quad (ii)\ If\ S = T \cup E\ with\ \emptyset \neq E \subset N\backslash T \in 2^N, then$$

$$u_T(S) = \frac{1}{|S|}\sum_{i\in S} u_T(S\setminus i)$$

and every player $i \in N\setminus T$ is $A-$ null player in the game u_T .

Theorem 2. *The family $\{u_T : \emptyset \neq T \in 2^N\}$ of games defined by Theorem 1 is a basis for the linear space $G_0(N)$.*

Theorem 3. *Define a function $\varphi^{sol}: G_0(N) \to (\mathbb{R}^n_+)^{2^N}$ by*

$$\varphi_i^{sol}(N, v) = \sum_{T\epsilon P_i(N)} \beta(|T|, |N|)A^v(T),$$

where $P_i(N)$ =$\{T \in 2^N | i \in T\}$ and $\beta(|T|, |N|) = \frac{(|T|-1)!(|N|-|T|)!}{|N|!}$. Then the function μ^{sol} is the unique solidarity value function on $G_0(N)$.

Proof. The proof proceeds exactly in the same way as that for the solidarity value function [2], namely $\varphi^{sol}(N, v)$and hence omitted.

1.1.2 Myerson Value for Network Games

A network on N is a set of unordered pairs of distinct players of N. A link between two distinct players $i \in N$ and $j \in N$ is an unordered pair, we will denote the link between i and j by $i{:}j = ij$. The set of all possible links on N is denoted by $g^N = \{i{:}j \mid i, j \in N, i \neq j\}$. The set of all possible networks defined on N is denoted by $G_N = \{g \mid g \subseteq g^N\}$. Two players $i \in N$ and $j \in N$ are said to connected in a coalition $S \subseteq N$ via the network g if $i = j$ or if there exists a sequence of players $i = i_0, i_1 \dots, i_k = j$, all in S such that each consecutive pair $i_{t,}i_{t+1} \in g$.

Definition 5. A pair $(v, g) \in G_0(N) \times G_N$ constitutes a game with communication graph structure or simply a network game on N. The sub-network of a network $g \in g^N$ with respect to set $T \subseteq N, T \neq \emptyset$ is the network $g_T \in g^T$ defined by $g_T = \{i{:}j \in g \mid i,j \in T\}$

Definition 6. Let $S \subseteq N$ and $g \in g^N$, then the set

$$S|_g = \{\{i| \ i \ and \ j \ are \ connected \ in \ S \ by \ g\}\}$$

represents the set of all connected components within the coalition S according to the connections defined by the network g. We denote $(S|_g)_i$ the component of S containing $i \in S$.

Definition 7. Let $v \in G_0(N)$ and $T \in 2^N$, the game v_T is called a sub-game and is defined by restricting v to coalition within T, that is,

$$v_T(S) = v(S), \quad \forall S \subseteq T.$$

Definition 8. Let $v : 2^N \to \mathbb{R}$ be a cooperative game and $g \in g^N$ be a network on N then a network game $v|_g$ is defined as

$$v|_g(S) = \textstyle\sum_{T\in S|_g} v(T), \ \forall S \subseteq N .$$

Definition 9. A function $\mu: G_0(N) \times G_N \to \mathbb{R}^n$ is said to be a Myerson value function for network if it satisfies the following axioms.

Axiom M1.($\boldsymbol{Component\ Efficiency}$) If for every network $(v,g) \in G_0(N) \times G_N$ on any player set N, for every $C \in N|_g$,

$$\sum_{i \in C} \mu_i(v,g) = v(C)$$

Axiom M2.($\boldsymbol{Myerson\ fairness}$) If for every network $(v,g) \in G_0(N) \times G_N$ on any player set N, for every $i\!:\!j \in g$,

$$\mu_i(v,g) - \mu_i(v, g_{-ij}) = \mu_j(v,g) - \mu_j(v, g_{-ij}), \qquad where\ g_{-ij} = g \backslash \{i\!:\!j\}.$$

Theorem 4. *Define a function* $\mu: G_0(N) \times G_N \to \mathbb{R}^n$ *by*

$$\mu_i(v,g) = \varphi_i^s(N, v|_g).$$

Then the function μ *is the unique Myerson value function for network games on* $G_0(N) \times G_N$.

Definition 10. A value function $\xi: G_0(N) \times G_N \to \mathbb{R}^n$ is said to be component balanced if for every network $(v,g) \in G_0(N) \times G_N$ on any player set N, for every component $C \in N|_g$,

$$\frac{\sum_{i \in C} \xi_i(v,g) - \xi_i(v_C, g_C)}{|C|} = \frac{\sum_{i \in N} \xi_i(v,g) - \xi_i(v_{(N|_g)_i}, g_{(N|_g)_i})}{|N|}.$$

Considering two components $C, C' \in N|_g$, this axiom implies that

$$\frac{\sum_{i \in C} \xi_i(v,g) - \xi_i(v_C, g_C)}{|C|} = \frac{\sum_{i \in N} \xi_i(v,g) - \xi_i(v_{C'}, g_{C'})}{|C'|}.$$

Definition 11. Let $v: 2^N \to \mathbb{R}$ be a cooperative game and $g \in g^N$ be a network on N then a network game $\overline{v|_g}$ is defined as

$$\overline{v|_g}(S) = \begin{cases} v|_g(S) = \sum_{T \in S|_g} v(T), & S \subset N, \\ v(N), & S = N, \end{cases}$$

Rane et al., (2011) introduced a value function as the new solution for network games that is efficiency, fairness and a new axiom refer to as component balancedness. Component balancedness can also be viewed as weaker form of component efficiency, because any value function for a network game that satisfies component efficiency will automatically satisfy component balancedness. We now proceed to formally define this value function as follows:

Theorem 5. *Define a function* $\psi: G_0(N) \times G_N \to \mathbb{R}^n$ *by*

$$\psi_i(v,g) = \varphi_i^s(N, \overline{v|_g}).$$

Then the function ψ *is the unique value function for network games on* $G_0(N) \times G_N$ *satisfying efficiency, fairness and component balancedness.*

2. Solidarity Value for Network Games

In this section, we revealed that the Myerson solidarity value function for network games does not satisfy component efficiency axioms, but it satisfies fairness. So, a new Solidarity value function is introduced using slightly different set of axioms: efficiency, fairness and the special axioms of component balancedness. We first present the uniqueness and existence of Myerson solidarity value function under its respective axioms. Subsequently, we define a new solidarity value function for network games and provide its characterization based on the above-mentioned axioms, with particular emphasis on the role of component balancedness.

Theorem 6. *Define a function* $\mu^{sol}: G_0(N) \times G_N \to \mathbb{R}^n$ *by*

$$\mu_i^{sol}(v,g) = \varphi_i^{sol}(N, v|_g).$$

Then the function μ *is the unique Myerson solidarity value function for network games on* $G_0(N) \times G_N$.

Proof. **Existence.** Along the line of *Theorem* of [8], fairness of Myerson solidarity value for network follows from symmetry of the solidarity value function and the connectedness structure imposed by g. Stability, however, requires v to be superadditive.

Uniqueness. The uniqueness of the Myerson solidarity value follows directly from the uniqueness of the Myerson solidarity value as established in *Theorem.* of [8].

Example 1. Let $N = \{1,2,3\}$, and consider the characteristic function v where $v(i) = 0,\ \forall\, i \in N, v(1,2) = v(2,3) = 10,$ *and* $v(1,3) = v(N) = 15$.

Let $g = \{1{:}2, 3{:}3\},\ so\ N|_g = \{\{1,2\},\{3\}\}.$

We know $\mu_i^{sol}(v,g) = \sum_{T \epsilon P_i(N)} \frac{(|T|-1)!(|N|-|T|)!}{|N|!} A^{v|_g}(T),$

Here $A^{v|_g}(\{i\}) = 0, A^{v|_g}(N) = \frac{20}{3}, A^{v|_g}(\{1,2\}) = 10, and\ A^{v|_g}(\{1,3\}) = A^{v|_g}(\{2,3\}) = 0$

Thus $\mu_1^{sol}(v,g) = \frac{35}{9}, \mu_2^{sol}(v,g) = \frac{35}{9}$ *and* $\mu_3^{sol}(v,g) = 0;$

$\sum_{i \in \{1,2\}} \mu_i^{sol}(v,g) = \frac{70}{9} \neq v(\{1,2\}).$

Definition 12. A function $\psi^{sol}: G_0(N) \times G_N \to \mathbb{R}^n$ is said to be a Solidarity value function for network if it satisfies the following axioms.

Axiom NS1.($\boldsymbol{Efficiency}$)If for every network $(v,g) \in G_0(N) \times G_N$ on any player set N,

$$\sum_{i \in N} \psi_i^{sol}(v,g) = v(N).$$

Axiom NS2.($\boldsymbol{fairness}$) If for every network $(v,g) \in G_0(N) \times G_N$ on any player set N, for every $i{:}j \in g$,

$$\psi_i^{sol}(v,g) - \psi_i^{sol}(v, g_{-ij}) = \psi_i^{sol}(v,g) - \psi_i^{sol}(v, g_{-ij}), \qquad where\ g_{-ij} = g \backslash \{i{:}j\}.$$

Axiom NS3.($\boldsymbol{component\ balancedness}$) If for every network $(v,g) \in G_0(N) \times G_N$ on any player set N, for every component $C \in N|_g$,

$$\frac{\sum_{i \in C} \psi_i^{sol}(v,g) - \psi_i^{sol}(v_C, g_C)}{|C|} = \frac{\sum_{i \in N} \psi_i^{sol}(v,g) - \psi_i^{sol}(v_{(N|_g)_i}, g_{(N|_g)_i})}{|N|}.$$

Theorem 7. *Define a function* $\psi^{sol}: G_0(N) \times G_N \to \mathbb{R}^n$ *by*

$$\psi_i^{sol}(v,g) = \varphi_i^{sol}(N, \overline{v|_g}).$$

Then the function ψ^{sol} *is the unique solidarity value function for network games satisfying efficiency, fairness and component balancedness on* $G_0(N) \times G_N$.

Proof. **Existence.** Since $\overline{v|_g}(N) = v(N)$, efficiency follows by efficiency of the Solidarity value. By Definition we have that $\overline{v|_g} = v|_g + w$, where $w \in G_0(N)$ is given by

$$w(S) = \begin{cases} 0, & S \subset N \\ (v(S) - v|_g(S)) \binom{|S|}{|N|}^{-1}, & S = N \end{cases}$$

As we know Myerson solidarity value is fair that is

$$\mu_i^{sol}(v,g) - \mu_i^{sol}(v, g\backslash\{i{:}j\}) = \mu_j^{sol}(v,g) - \mu_j^{sol}(v, g\backslash\{i{:}j\}),$$

Hence,

$\psi_i^{sol}(v,g) - \psi_i^{sol}(v,g\backslash\{i{:}j\}) = \varphi_i^{sol}(\overline{v|_g}) - \varphi_i^{sol}(\overline{v|_{g\backslash\{i:j\}}})$

$= \mu_i^{sol}(v,g) + \frac{v(N)-v|_g(N)}{n} - \mu_i^{sol}(v,g\backslash\{i{:}j\}) - \frac{v(N)-v|_{g\backslash\{i:j\}}(N)}{n}$

$= \mu_j^{sol}(v,g) - \mu_j^{sol}(v,g\backslash\{i{:}j\}) - \frac{v|_g(N)-v|_{g\backslash\{i:j\}}(N)}{n}$

$= \psi_j^{sol}(v,g) - \psi_j^{sol}(v,g\backslash\{i{:}j\})$

Hence ψ^{sol} satisfies fairness.

Along lines of *Theorem 3.1* of [7] we can easily obtained value function for network games satisfies component balancedness.

Uniqueness. The uniqueness of the solidarity value follows directly from the uniqueness of the Shapley value as established in *Theorem 3.2* of [7].

Example 2. (cf. Example 1) Let $N = \{1,2,3\}$, and consider the characteristic function v where $v(i) = 0,\ \forall\, i \in N, v(1{,}2) = v(2{,}3) = 10,\ and\ v(1{,}3) = v(N) = 15$.

Let $g = \{1{:}2, 3{:}3\},\ so\ N|_g = \{\{1,2\},\{3\}\}$.

We know,

$\psi_i^{sol}(v,g) = \sum_{T\epsilon P_i(N)} \frac{(|T|-1)!(|N|-|T|)!}{|N|!} A^{\overline{v|_g}}(T), \quad A^{\overline{v|_g}}(T) = \frac{1}{|T|}\sum_{k\in T}[\overline{v|_g}(T) - \overline{v|_g}(T \setminus k)];$

Here $A^{\overline{v|_g}}(\{i\}) = 0, A^{\overline{v|_g}}(N) = 15, A^{\overline{v|_g}}(\{1,2\}) = 10, and\ A^{v|_g}(\{1,3\}) = A^{v|_g}(\{2,3\}) = 0$

Thus $\psi_1^{sol}(v,g) = 5\frac{10}{18}, \psi_2^{sol}(v,g) = 5\frac{10}{18}\ and\ \psi_2^{sol}(v,g) = 3\frac{16}{18}$.

3. Conclusion

The introduction of Myerson solidarity value function and the solidarity value function offers a new perspective on allocation rules in network games. While the Myerson solidarity value does not fulfil the component efficiency axiom, the solidarity value function satisfies weaker yet meaningful axiom known as component balancedness. Since every component efficient solidarity value function for network games is also component balanced, the solidarity value function for network games satisfies fairness, components balancedness and overall efficiency. Future investigations into additional axioms may shed more light on the mathematical structure and potential applications of both functions.

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