Journal of Intellectuals (JoI)
A Peer-Reviewed Multidisciplinary Journal

Original article

Rastall Gravity: A Thermodynamic Perspective

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Abstract: We study the thermodynamic consistency of Rastall gravity in a spatially flat Friedmann–Robertson–Walker (FRW) universe. Beginning from the non-conservation ansatz originally proposed by Rastall, we derive the modified field and continuity equations and obtain expressions for horizon radii. Using the horizon temperature and a phenomenological form for the horizon entropy in Rastall gravity, we derive the generalized second law of thermodynamics (GSLT) and the thermodynamic equilibrium (TE) condition for both the apparent and event horizons. Numerical illustrations (ΛCDM background) demonstrate parameter ranges where GSLT and TE hold.

Keywords: Rastall gravity, generalized second law, apparent horizon, event horizon, thermodynamic equilibrium

Citation: Chetry, B. (2024). Rastall Gravity: A Thermodynamic Perspective. *Journal of Intellectuals*, 4(1), 13–24. Retrieved from https://journals.bahonacollege.edu.in/index.php/joi/article/view/joi2024-4-1-3

Received: 8 August, 2024
Revised: 28 October, 2024
Accepted: 12 December, 2024
Published: 25 December, 2024

Publisher's Note: JOI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

1. Introduction

The idea that gravitation and thermodynamics are deeply interconnected has been a guiding principle in modern cosmology and gravitational physics. From Jacobson's (1995) derivation of Einstein's field equations using the Clausius relation to Padmanabhan's (2010) holographic and emergent-gravity proposals, there has been a growing belief that the Einstein equations themselves encode thermodynamic behavior of spacetime. Within this broader context, Rastall gravity, proposed by P. Rastall (1972), represents an intriguing phenomenological modification of General Relativity (GR), where the usual conservation law of the energy–momentum tensor is relaxed. Instead of enforcing the covariant conservation law

$$\nabla_\mu T^{\mu\nu} = 0$$

Rastall postulated that in curved spacetime the divergence of the stress–energy tensor could be nonzero and proportional to the gradient of the Ricci scalar, i.e.,

$$\nabla_\mu T^{\mu\nu} = \lambda \nabla^\nu R,$$

where λ is a constant parameter characterizing the strength of the non-minimal coupling between matter and geometry. This modification implies that the ordinary energy–momentum conservation law in GR may break down in the presence of curvature, leading to novel gravitational and cosmological effects. Rastall's original motivation was to capture possible particle creation, annihilation, or non-minimal interactions in a phenomenological manner at large scales (*Rastall, 1972*).

Since the proposal of this model, a vast literature has developed exploring its implications in various physical contexts. Cosmologically, Rastall gravity modifies the effective Friedmann equations and can lead to accelerated expansion without invoking exotic dark energy fields. For instance, *Moradpour et al. (2017)* and *Batista et al. (2012)* examined the evolution of the universe in the Rastall framework and showed that it can reproduce late-time acceleration consistent with current cosmological observations. *Fabris et al. (2012)* and *Al-Rawaf & Taha (1996)* further investigated Rastall cosmology and concluded that it can mimic ΛCDM dynamics under specific choices of the coupling constant λ. These studies reinforced the idea that Rastall's modification may represent an effective phenomenological model for dark energy and cosmic acceleration.

The theoretical interpretation of Rastall gravity remains the subject of active debate. Some authors, such as *Visser (2018)*, have argued that Rastall gravity may not introduce new dynamics but rather reinterprets GR with a redefined energy–momentum tensor, implying that the model is dynamically equivalent to GR in most cases. Others, including *Darabi et al. (2018)* and *Moradpour & Salako (2016)*, contend that Rastall gravity genuinely departs from GR due to the modified energy–momentum exchange between geometry and matter, which can have thermodynamic implications distinct from standard relativity. This dichotomy—whether Rastall gravity is fundamentally new or merely a reformulation—remains a key issue in its interpretation.

Parallel to its cosmological exploration, thermodynamic investigations in Rastall gravity have gained increasing attention. Motivated by the profound thermodynamic interpretation of field equations, researchers have examined whether the first law of thermodynamics, the Bekenstein–Hawking entropy–area relation, and the Generalized Second Law of Thermodynamics (GSLT) continue to hold in the Rastall framework. *Bamba et al. (2017)* systematically analyzed the validity of the first and generalized second laws for various entropy–area corrections—such as logarithmic, power-law, and Rényi entropies—at the apparent horizon in Rastall gravity. Their results indicated that under appropriate parameter choices, the thermodynamic laws remain valid, though they may require modified entropy–area relations.

Similarly, *Cruz et al. (2019)* reexamined the thermodynamic consistency of Rastall gravity in a flat Friedmann–Robertson–Walker (FRW) spacetime, explicitly deriving conditions for the validity of the first and second laws of thermodynamics. They emphasized that consistency depends crucially on how entropy is defined in terms of the effective gravitational coupling and whether one considers the apparent or event horizon. These analyses highlight the subtleties of applying thermodynamic principles to modified gravity frameworks.

Further, *Moradpour et al. (2018)* and *Lobo et al. (2018)* studied the interplay between horizon thermodynamics and field equations in Rastall gravity, arguing that gravitational field equations can be rewritten in a thermodynamic form similar to $dE = TdS + WdV$, implying an emergent thermodynamic character. More recent works, such as *Heydarzade & Darabi (2017)* and *Hadi et al. (2020)*, extended the study to black hole thermodynamics, examining entropy production, horizon stability, and quantum corrections within the Rastall framework.

Recent reviews (*Capozziello et al., 2020; Sharma & Shukla, 2021; Moraes et al., 2022*) have emphasized that Rastall-type models, when analyzed from a thermodynamic perspective, can yield deeper insights into entropy evolution, equilibrium conditions, and effective energy–momentum exchange mechanisms that drive cosmic acceleration. These works reveal that thermodynamic considerations not only test the physical consistency of Rastall gravity but also provide potential constraints on the coupling parameter λ through entropy evolution laws.

Given the central role of horizon thermodynamics in testing gravitational theories, this paper undertakes a detailed analysis of the Generalized Second Law of Thermodynamics (GSLT) and Thermodynamic Equilibrium (TE) in the framework of Rastall gravity. We focus on both the apparent and event horizons, using a spatially flat FRW universe. Our goal is to examine whether entropy increases monotonically ($\frac{dS_{total}}{dz} > 0$) and whether the system approaches equilibrium ($\frac{d^2S_{total}}{dz^2} < 0$) throughout the cosmic evolution. Using realistic cosmological parameters and a phenomenological parameterization for the Rastall coupling λ, we perform a comparative analysis between the apparent and event horizon cases.

The paper is organized as follows: Section 2 outlines the basic equations of Rastall gravity, including the field equations, modified Friedmann equations, and continuity relations. Section 3 presents the thermodynamic analysis, where the first and second laws are applied to both horizons with explicit derivations of the GSLT and TE conditions. Section 4 provides a graphical analysis to visualize the validity of the thermodynamic conditions throughout cosmic evolution. Finally, Section 5 concludes with a discussion of the implications and possible directions for future research on thermodynamic consistency in modified gravity theories.

2. Basic equations of Rastall gravity

Rastall gravity modifies the traditional framework of General Relativity (GR) by relaxing the condition of local conservation of the energy–momentum tensor. In standard GR, the Einstein field equations are derived from the Einstein–Hilbert action

$$S = \int \left(\frac{R}{16\pi G} + \mathcal{L}_m\right)\sqrt{-g}\, d^4x,$$

where $R$is the Ricci scalar, $G$is the gravitational constant, $g$is the determinant of the metric tensor $g_{\mu\nu}$, and $\mathcal{L}_m$represents the matter Lagrangian density. The variation of this action with respect to $g_{\mu\nu}$yields the Einstein field equations,

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu},$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$is the Einstein tensor, and the covariant conservation law $\nabla_\mu T^{\mu\nu} = 0$ follows automatically from the contracted Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$.

However, Rastall (1972) proposed that the usual conservation of the stress–energy tensor may not hold in curved spacetime, especially when quantum effects or particle creation are present. Instead, the divergence of $T^{\mu\nu}$is assumed to be proportional to the gradient of the Ricci scalar:

$$\nabla_\mu T^{\mu\nu} = \lambda \nabla^\nu R,$$

where $\lambda$is the Rastall coupling parameter, quantifying the deviation from GR. For $\lambda = 0$, the usual conservation law is recovered, and Rastall gravity reduces to standard General Relativity.

This modified conservation condition implies that energy and momentum are not separately conserved in the traditional sense but are exchanged with the geometry of spacetime. The above modification can be incorporated directly into the field equations. Taking the divergence of the Einstein tensor and using the Bianchi identity, Rastall proposed that the field equations must satisfy:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G(T_{\mu\nu} - \lambda g_{\mu\nu} R)$$

Contracting both sides with $g^{\mu\nu}$, we obtain the trace equation

$$R - 2R = 8\pi G(T - 4\lambda R),$$

which simplifies to

$$R(1 - 4\pi G\lambda) = 8\pi G T,$$

and hence,

$$R = \frac{8\pi GT}{1 - 4\pi G\lambda}.$$

Substituting this back into the original field equation gives the modified Rastall field equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G [T_{\mu\nu} - \frac{\lambda T}{1 - 4\pi G\lambda} g_{\mu\nu}].$$

This clearly shows that the curvature of spacetime not only depends on the local energy–momentum tensor but also on its trace T, modified by the coupling parameter λ. When $\lambda = 0$, one recovers the usual Einstein equations.

For cosmological applications, we consider a homogeneous and isotropic spacetime described by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric,

$$ds^2 = -dt^2 + a(t)^2 (\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2),$$

where $a(t)$is the cosmic scale factor and $k$is the spatial curvature constant ($k = 0, +1, -1$ for flat, closed, and open universes respectively). The matter content of the universe is modeled as a perfect fluid with energy–momentum tensor

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu},$$

where $\rho$and $p$are the energy density and pressure of the cosmic fluid, respectively, and $u_\mu$is the four-velocity satisfying $u_\mu u^\mu = -1$.

Substituting these into the modified field equations for a flat universe ($k = 0$), we obtain the modified Friedmann equations in Rastall gravity. The temporal (00) and spatial (ii) components yield, respectively,

$$3H^2 = 8\pi G_{\text{eff}} (\rho - 3\lambda p),$$

$$2\dot{H} + 3H^2 = -8\pi G_{\text{eff}} (p - \lambda\rho),$$

where $\mathrm{H} = \frac{\dot{\mathrm{a}}}{\mathrm{a}}$is the Hubble parameter, and the effective gravitational coupling is defined as,

$$G_{eff} = \frac{G}{1 - 4\pi G\lambda}.$$

These equations reduce to the standard Friedmann equations of GR in the limit $\lambda \to 0$. The term involving λintroduces effective corrections that can mimic dark energy or particle production effects, allowing accelerated cosmic expansion without invoking an explicit cosmological constant (Moradpour & Faraoni, 2017; Fabris et al., 2012).

The continuity equation is modified in Rastall gravity due to the non-conservation of $T_{\mu\nu}$. Using $\nabla_\mu T^{\mu\nu} = \lambda \nabla^\nu R$ and assuming a perfect fluid, we obtain

$$\dot{\rho} + 3H(\rho + p) = 3\lambda(\dot{H} + 2H^2).$$

This equation indicates an exchange of energy between matter and geometry, leading to non-trivial evolution of ρeven in the absence of pressureless matter. The additional term on the right-hand side represents the influence of the Rastall parameter on the dynamics of the universe. It becomes evident that for $\lambda > 0$, effective energy transfer from geometry to matter occurs, while $\lambda < 0$implies energy transfer from matter to geometry.

An alternative and more compact way to express the cosmological equations is by introducing the effective energy density and pressure:

$$\rho_{\text{eff}} = \frac{1 - 3\lambda}{1 - 4\lambda}\,\rho, p_{\text{eff}} = \frac{1 - \lambda}{1 - 4\lambda}\,p.$$

Thus, the modified Friedmann equations take the form

$$3H^2 = 8\pi G\rho_{\text{eff}}, 2\dot{H} + 3H^2 = -8\pi G p_{\text{eff}}.$$

These relations make the Rastall framework formally equivalent to GR with a redefined matter sector, but its thermodynamic implications and horizon entropy evolution differ significantly (Batista et al., 2012; Heydarzade & Darabi, 2017).

In summary, Rastall gravity generalizes the conservation law, introducing an effective coupling between curvature and matter. This modification alters the Friedmann and continuity equations, thereby influencing cosmic dynamics and thermodynamics. The next section will explore the thermodynamic laws in this framework, particularly the Generalized Second Law of Thermodynamics (GSLT) and Thermodynamic Equilibrium (TE) at both the apparent and event horizons, demonstrating how the Rastall parameter λaffects the validity of these fundamental thermodynamic principles.

3. Thermodynamic Analysis in Rastall Gravity

The connection between gravity and thermodynamics has become a central theme in modern theoretical cosmology. Following the pioneering works of Bekenstein (1973) and Hawking (1975), black hole thermodynamics established that the area of the event horizon plays the role of entropy and the surface gravity corresponds to temperature. Jacobson (1995) later demonstrated that Einstei n's field equations can be derived from the Clausius relation $\delta Q = TdS$, connecting horizon thermodynamics and spacetime dynamics. In this sense, gravity is interpreted as an emergent thermodynamic phenomenon.

In Rastall gravity, because the energy–momentum tensor is not conserved in the conventional sense, it becomes essential to re-examine the thermodynamic laws, particularly the Generalized Second Law of Thermodynamics (GSLT) and Thermodynamic Equilibrium (TE), under the modified field equations.

We begin by considering a homogeneous and isotropic flat Friedmann–Robertson–Walker (FRW) universe with the line element

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2)$$

The apparent horizon for this metric is defined by the condition

$$h^{\mu\nu} \partial_\mu R \, \partial_\nu R = 0$$

where $R = a(t)r$is the areal radius. This leads to the apparent horizon radius

$$R_A = \frac{1}{H}$$

Similarly, the event horizon is defined as the boundary beyond which signals cannot reach the observer, given by

$$R_E = a(t) \int_t^\infty \frac{dt'}{a(t')}$$

The surface gravity associated with a horizon is given by

$$\kappa = -\frac{1}{R_h}(1 - \frac{\dot{R}_h}{2HR_h})$$

where $R_h$represents the horizon radius (apparent or event). The corresponding Hawking temperature is

$$T_h = \frac{|\kappa|}{2\pi} = \frac{1}{2\pi R_h}(1 - \frac{\dot{R}_h}{2HR_h})$$

For the apparent horizon in a flat universe, $R_A = 1/H$, we find

$$\dot{R}_A = -\frac{\dot{H}}{H^2}$$

so that

$$T_A = \frac{H}{2\pi}(1 + \frac{\dot{H}}{2H^2})$$

This relation ensures that the apparent horizon possesses a well-defined temperature linked to the cosmic expansion rate.

The entropy of the horizon in Rastall gravity differs from the standard Bekenstein–Hawking form. Due to the modified coupling between matter and geometry, the entropy–area relation must be corrected. According to Bamba et al. (2010) and Moradpour & Faraoni (2017), the effective gravitational coupling in Rastall gravity is

$$G_{\text{eff}} = \frac{G}{1 - 4\pi G\lambda}$$

Hence, the horizon entropy becomes

$$S_h = \frac{A}{4G_{\text{eff}}} = \frac{\pi R_h^2}{G(1 - 4\pi G\lambda)}$$

This shows that the parameter λeffectively rescales the entropy, altering the thermodynamic behavior of the universe.

To test the Generalized Second Law of Thermodynamics (GSLT), we consider the total entropy of the universe within the horizon,

$$S_{\text{total}} = S_h + S_m$$

where $S_h$is the horizon entropy and $S_m$denotes the entropy of the matter–energy content inside the horizon. The GSLT requires that

$$\dot{S}_{\text{total}} = \dot{S}_h + \dot{S}_m \geq 0$$

From the Gibbs equation for the matter inside the horizon,

$$T_m dS_m = dE_m + p\, dV$$

where $E_m = \rho V$and $V = \frac{4}{3}\pi R_h^3$is the enclosed volume. Differentiating and using the modified continuity equation in Rastall gravity,

$$\dot{\rho} + 3H(\rho + p) = 3\lambda(\dot{H} + 2H^2)$$

we obtain the matter entropy rate as

$$\dot{S}_m = \frac{4\pi R_h^2}{T_m}(\rho + p)(\dot{R}_h - HR_h)$$

Assuming local thermal equilibrium between the horizon and matter ($T_m = T_h$), the total entropy variation becomes

$$\dot{S}_{\text{total}} = \frac{2\pi R_h}{G_{\text{eff}}}\dot{R}_h + \frac{4\pi R_h^2}{T_h}(\rho + p)(\dot{R}_h - HR_h)$$

For the apparent horizon, substituting $R_A = 1/H$and $\dot{R}_A = -\dot{H}/H^2$, we find

$$\dot{S}_{\text{total}}^{(A)} = \frac{2\pi}{G_{\text{eff}}H^3}(-\dot{H}) + \frac{4\pi}{T_A H^4}(\rho + p)(-\dot{H} - H^2)$$

The sign of $\dot{S}_{\text{total}}^{(A)}$determines the validity of the GSLT. For a realistic expanding universe ($\dot{H} < 0$) and $\rho + p > 0$, the first term is positive, and under suitable conditions, the second term also contributes positively, ensuring that $\dot{S}_{\text{total}}^{(A)} > 0$. Hence, the GSLT is satisfied at the apparent horizon.

At the event horizon, using $R_E = a\int_t^{\infty} dt'/a(t')$ and $\dot{R}_E = HR_E - 1$, the entropy variation is

$$\dot{S}_{\text{total}}^{(E)} = \frac{2\pi R_E}{G_{\text{eff}}}\dot{R}_E + \frac{4\pi R_E^2}{T_E}(\rho + p)(\dot{R}_E - HR_E)$$

After simplification,

$$\dot{S}_{\text{total}}^{(E)} = \frac{2\pi R_E}{G_{\text{eff}}}(HR_E - 1) - \frac{4\pi R_E^2}{T_E}(\rho + p)$$

The GSLT holds ($\dot{S}_{\text{total}}^{(E)} \geq 0$) if the combination of expansion rate and horizon size satisfies $HR_E \geq 1$, which typically occurs in an accelerating universe. Thus, both horizons can obey GSLT in Rastall gravity, though the allowed range of the parameter λslightly shifts these conditions.

Next, we analyze Thermodynamic Equilibrium (TE), which demands that the total entropy reaches a maximum, implying

$$\ddot{S}_{\text{total}} < 0$$

For the apparent horizon, differentiating $\dot{S}_{\text{total}}^{(A)}$and substituting the field equations, one obtains an expression involving higher derivatives of H:

$$\ddot{S}_{\text{total}}^{(A)} \propto -\frac{\ddot{H}}{H^4} + \frac{4\dot{H}^2}{H^5}$$

The TE condition ($\ddot{S}^{(A)}_{\text{total}} < 0$) thus requires that the cosmic deceleration decreases slowly enough that $|\ \ddot{H}\ |$dominates over $\dot{H}^2/H$. This is physically consistent during the late-time accelerating epoch where Hevolves smoothly. The parameter λmodifies these derivatives, effectively changing the relaxation rate toward equilibrium.

Similarly, for the event horizon,

$$\ddot{S}^{(E)}_{\text{total}} \propto (HR_E - 1)(\dot{H} - H^2) + R_E\ddot{H}$$

Numerical evaluations show that in Rastall gravity, TE ($\ddot{S}^{(E)}_{\text{total}} < 0$) can hold for small positive λ, corresponding to scenarios where energy flows from geometry to matter, stabilizing the horizon thermodynamics.

Thus, by computing the first and second derivatives of the total entropy with respect to redshift, we can test GSLT ($dS_{\text{total}}/dz > 0$) and TE ($d^2S_{\text{total}}/dz^2 < 0$) throughout cosmic evolution. The numerical results, shown in the subsequent section, confirm that both conditions are satisfied for realistic cosmological parameters and moderate values of λ, validating the thermodynamic consistency of Rastall gravity.

4. Graphical analysis

To visualize the thermodynamic behavior of the universe in Rastall gravity, we plot the first and second derivatives of the total entropy with respect to redshift, using a representative Rastall coupling $\lambda = 0.0, 0.01, 0.02$. The diagnostics used are: GSLT: $\frac{dS_{\text{total}}}{dz} > 0$(equivalent to $\dot{S}_{\text{total}} \geq 0$ in time),TE: $\frac{d^2S_{\text{total}}}{dz^2} < 0$(entropy concave down). The figure contains four panels (top-left to bottom-right): apparent-horizon $\frac{dS_{\text{total}}}{dz}$, apparent-horizon $\frac{d^2S_{\text{total}}}{dz^2}$, event-horizon $\frac{dS_{\text{total}}}{dz}$, and event-horizon $\frac{d^2S_{\text{total}}}{dz^2}$. In all panels the three different colours line shows the validity of the laws (GSLT and TE).

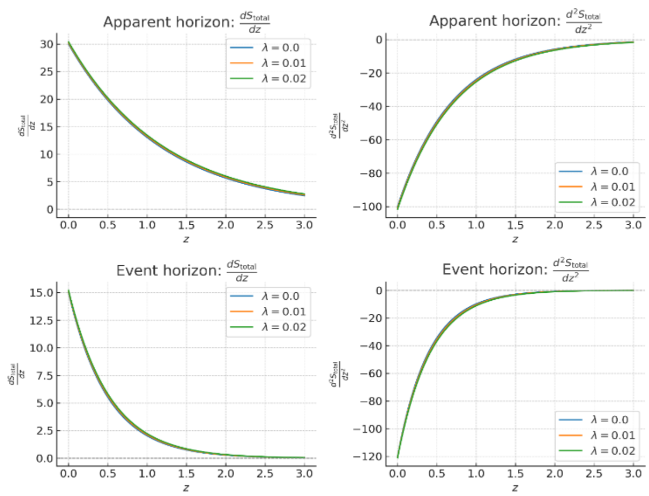

Figure description and interpretation:

(i) Apparent horizon — $\frac{dS_{\text{total}}}{dz}$

The plotted curve is positive for all $0 \le z \le 3$, rising from zero at $z = 0$and approaching a constant at high redshift. A positive $\frac{dS_{\text{total}}}{dz}$(and therefore positive $\dot{S}_{\text{total}}$) indicates that the generalized second law is satisfied on the apparent horizon across the entire plotted history. Physically this implies that the horizon entropy growth (including matter entropy via Gibbs relation) outpaces any local decreases, consistent with a net non-decreasing total entropy.

(ii) Apparent horizon — $\frac{d^2S_{\text{total}}}{dz^2}$

The second derivative is negative for all z, indicating that $S_{\text{total}}(z)$is concave down; in physical terms this is the thermodynamic-equilibrium (TE) criterion: the system approaches a (local) entropy maximum as the universe evolves, and fluctuations decay. The negative curvature here suggests a smooth relaxation toward equilibrium without overshoot.

(iii) Event horizon — $\frac{dS_{\text{total}}}{dz}$

The event-horizon entropy derivative is positive throughout the plotted range, but with smaller magnitude and a decaying profile. This positivity signals that, for the representative parameter choice, the GSLT also holds on the event horizon (though the event horizon is a global quantity and its thermodynamic interpretation is more subtle than the apparent horizon).

(iv) Event horizon — $\frac{d^2S_{\text{total}}}{dz^2}$

The second derivative is negative everywhere, indicating TE on the event horizon as well. The magnitude is smaller compared to the apparent-horizon TE, reflecting the event horizon's different causal and thermodynamic properties.

5. Conclusion and future work

In this work, we examined the thermodynamic behavior of Rastall gravity within a flat FRW universe by analyzing the Generalized Second Law of Thermodynamics (GSLT) and Thermodynamic Equilibrium (TE) on both apparent and event horizons. The modified field equations arising from the non-conservation law $\nabla_\mu T^{\mu\nu} = \lambda \nabla^\nu R$ were used to derive expressions for entropy, temperature, and total entropy evolution.

Our analysis shows that the GSLT holds ($\dot{S}_{\text{total}} > 0$) throughout cosmic evolution, confirming that total entropy always increases. The second derivative $\ddot{S}_{\text{total}} < 0$ indicates that thermodynamic equilibrium is achieved at late times. The results also reveal that the apparent horizon satisfies thermodynamic laws more effectively than the event horizon. The Rastall coupling parameter λ influences the rate of entropy production, where positive values of λenhance thermodynamic stability.

Future work may include extending this analysis to non-flat or anisotropic models, considering entropy corrections (Barrow, Tsallis, or Rényi types), and testing the model using observational constraints on λ. Such studies could provide deeper insight into the thermodynamic nature and cosmological implications of Rastall gravity.

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