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Solidarity Value for Fuzzy Cooperative Games with Multilinear extension form

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Abstract: In this paper, we extend the classical Solidarity value of cooperative games to the framework of fuzzy cooperative games. Using multilinear extension form, we formulate an explicit expression for the Solidarity value in the fuzzy setting and establish its existence and uniqueness. Several fundamental properties of the proposed value are investigated.

Keywords: Cooperative games, Fuzzy Cooperative games, Solidarity value, Multilinear extension form value, Multilinear extension form

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1. Introduction

The Solidarity value [2] is an important solution concept in cooperative game theory, designed to allocate the total worth of a coalition among players while reflecting a notion of collective responsibility. Similar to Shapley value [4], it assigns a payoff vector to players based on their contributions, but it emphasizes solidarity among participants. Classical cooperative games models assume that coalitions are crisp, meaning that each player either fully participates (membership 1) or does not participate at all (membership 0).

However, in many real-world situations involve partial participation rather than complete commitment. To capture such cases fuzzy cooperative games were introduced. Aubin [3] the concept of fuzzy cooperative games to model situations in which players may belong to coalitions with varying degrees of membership level between 0 and 1. In this framework, the membership degree represents the extent to which a player contributes resources or effort to a coalition.

Developing appropriate solution concepts for fuzzy cooperative games remains an important research direction. Although several extensions of classical value have been studied in [5-8]. the solidarity value in context of fuzzy cooperative games particularly via the multilinear extension approach has received relatively limited attention.

Motivated by this gap, the present paper extends the classical Solidarity value to fuzzy cooperative games using multilinear extension framework.

1.1 PRELIMINERIES

1.1.1 *CLASSICAL COOPERATIVE GAMES AND SOLIDARITY VALUE*

Let the player set be given $N = \{1, 2, \ldots, n\}$. Let 2^N denote the set of all coalitions obtained from the player set. A TU cooperative game or simply a cooperative game is the pair (N, v) where where $N = \{1, 2, \ldots, n\}$ is a set of n players, called the grand coalition and v is the function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. If no ambiguity about the player set N arises, we denote a TU game by its characteristic function v only. Let $G_0(N)$ denote the class of all TU games with player set N. Recall that a solution of a TU-game with n players is an n-dimensional vector representing a distribution of payoffs. A value function is an allocation rule on a subset C of $G_0(N)$ is a function that assigns a solution to any game in C.

Definition 1. Let $T_0 \in 2^N$ and $v \in G_0(N)$ the quantity

$$\Delta^v(T_0) = \frac{1}{|T_0|} \sum_{k \in T_0} [v(T_0) - v(T_0 \setminus k)]$$

is called the average marginal contribution of a player of the coalition T.
Definition 2. Given a game $v \in G_0(N)$ Player $i \in N$ is called a $\Delta -$ null player if $\Delta^v(T_0) = 0$ for every coalition $T_0 \subseteq N$ containing i.
Definition 3. A function $f_i : G_0(N) \to (\mathbb{R}^n_+)^{2^N}$ is said to be a solidarity value function on $G_0(N)$ if it satisfies the following four axioms.
Axiom S1.(*Efficiency*)If $v \in G_0(N)$

$$\sum_{i \in W_0} f_i(W_0, v) = v(W_0)$$

Axiom S2.($\Delta -$ *null player*) If $v \in G_0(N)$ and $i \in W_0 \in 2^N$ is a $\Delta -$ null player, that is $\Delta^v(T) = 0$ then

$$f_i(W_0, v) = 0 \quad \forall \ i \in W_0 \subset N,$$

Axiom S3.(*Symmetry*) If $v \in G_0(N)$ and $i, j \in N$ are symmetric i.e., $v(S \cup i) = v(S \cup j)$ holds for any $S \subseteq W_0 \in 2^{N \setminus \{i,j\}}$, then

$$f_i(W_0, v) = f_j(W_0, v),$$

Axiom S4.(*Additivity*)For $v_1, v_2 \in G_0(N)$, define $v_1 + v_2 \in G_0(N)$ by $(v_1 + v_2)(W_0) = v_1(W_0) + v_2(W_0)$ for each $S \subseteq W_0 \in 2^N$. If $v_1, v_2 \in G_0(N)$ then

$$f_i(W_0, v_1 + v_2) = f_i(W_0, v_1) + f_i(W_0, v_2)$$

Theorem 1.*[2] Define a function$f_i : G_0(N) \to (\mathbb{R}^n_+)^{2^N}$ by*

$$f_i(W_0, v) = \sum_{T \epsilon P_i(N)} \beta(|T_0|, |W_0|) \Delta^v(T_0),$$

where $P_i(N)$ =$\{W_0 \in 2^N | i \in W_0\}$ and $\beta(|T_0|, |N|) = \frac{(|T_0| - 1)!(|W_0| - |T_0|)!}{|W_0|!}$. Then the function f is the unique solidarity value function on $G_0(N)$.

Proof. The proof proceeds exactly in the same way as that for the solidarity value function [2], namely f and hence omitted.

1.1.2 *FUZZY COOPERATIVE GAMES*

A fuzzy coalition is a fuzzy subset of $N = \{1, 2, \dots, n\}$, which is identified with a characteristic function from N to [0,1]. Let $L(N)$ be the set of all fuzzy coalitions in N. For a fuzzy coalition $S \in L(N)$ and player $i \in N$, $S(i)$ represents the membership grade of i in S. The empty fuzzy coalition denoted by $\emptyset$ is one where all the players provide zero membership. If no ambiguity arises we use the same notations to represent crisp and fuzzy coalitions as crisp coalitions are special fuzzy coalitions with memberships 0 or 1.

The support of a fuzzy coalition S is denoted by Supp S $= \{i \in N \mid S(i) > 0\}$. We use the notation $S \subseteq T$ if and only if $S(i) \leq T(i)$ for all $i \in N$. Let $\vee$ and $\wedge$ respectively represent the maximum and the minimum operators. The union and intersections of fuzzy coalitions S and T given by $S \cup T$ and $S \cap T$ are defined as $(S \cup T)(i)$ = $(S \vee T)(i)$ and $(S \cap T)(i)$ = $(S \wedge T)(i)$ respectively for each $i \in N$.

Definition 4. A TU game with fuzzy coalitions or simply a fuzzy cooperative game is a pair (U, v) where $U \in L(N)$ and $v : L(N) \to \mathbb{R}$ is a set function, satisfying $v(\emptyset) = 0$.

Let $FG(N)$ denote the class of TU fuzzy games with player set N. Now we define fuzzy cooperative games in Multilinear extension form due to [5].

Definition 5. Let $S, T \in L(N)$ A fuzzy cooperative game $v \in FG(N)$ is called fuzzy convex if $v(S \vee T) + v(S \wedge T) \geq v(T) + v(S)$.

Definition 6. Let $U \in L(N)$, $S \subseteq U$ and $v \in FG(N)$. A player $i \in Supp\ U$ but $i \notin Supp\ S$ is called a fuzzy null player for v in U if $v(S \vee U(i)) = v(S)$.

Definition 7. Let $U \in L(N)$, $T \subseteq U$ and $v \in FG(N)$. A fuzzy coalition S is called fuzzy carrier for v in U if $v(S \wedge T) = v(T)$.

Definition 8. The pair (U, v) is called a fuzzy cooperative game in multilinear extension form if for every fuzzy coalition S $\subseteq U$,

$$v(S) = \sum_{T_0 \subseteq \text{Supp S}} \left\{\prod_{i \in T_0} U(i) \prod_{i \in \text{Supp U} \setminus T_0} (1 - U(i))\right\} v(T_0) .$$

We use $\mathrm{FG}_M(N)$ to denote set of this kind of fuzzy cooperative games on $U \in L(N)$.

Definition 9. A function $\psi_i : L(U) \to (\mathbb{R})^{L(U)}$ is said to be a Shapley value function on $\mathrm{FG}_M(U)$. if it satisfies the following four axioms.

Axiom FS1.(*Efficiency*) If $v \in \mathrm{FG}_M(U)$.

$$\sum_{i \in Supp\ U} \psi_i(U, v) = v(U)$$

Axiom FS2.(*null player*) If $v \in \mathrm{FG}_M(U)$ and $i \in Supp\ U$ but $i \notin Supp\ S$ is a nullplayer, that is $v(S \vee U(i)) = v(S)$, $\forall\ S \subseteq U$. then

$$\psi_i(U, v) = 0$$

Axiom FS3.(*Symmetry*) Let $v \in \mathrm{FG}_M(U)$ for any $i, j \in Supp\ U$ and any $S \subseteq U$ with $i, j \notin Supp\ S$ are symmetric i.e., $v(S \vee U(i)) = v(S \vee U(j))$, then

$$\psi_i(U, v) = \psi_j(U, v),$$

Axiom FS4.(*Additivity*) For $v_1, v_2 \in \mathrm{FG}_M(U)$, define $v_1 + v_2 \in \mathrm{FG}_M(U)$ by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for each $S \subseteq U$, then

$$\psi_i(U, v_1 + v_2) = \psi_i(U, v_1) + \psi_i(U, v_2), \qquad \forall\ i \in Supp\ U$$

Theorem 2.*[7]* Define a function $\psi : \mathrm{FG}_M(N) \to (\mathbb{R})^{\mathrm{FG}_M(N)}$ by

$\psi_i(U,v) = \sum_{T_0 \subseteq \text{Supp U} \backslash \{i\}} \frac{(|T_0|)!(|Supp\ U| - |T_0|-1)!}{|Supp\ U|!} M^v(T_0)$, $\forall\ i \in Supp\ U$.

Where

$$M^v(T_0) = \left\{ \sum_{S_0 \subseteq T_0 \cup \{i\}} \prod_{j \in S_0} U(j) \prod_{j \in \text{Supp U} \backslash S_0} (1 - U(j)) \right\} v(S_0) - \left\{ \sum_{S_0 \subseteq T_0} \prod_{j \in S_0} U(j) \prod_{j \in \text{Supp U} \backslash S_0} (1 - U(j)) \right\} v(S_0)$$

Then the function ψ is the unique Shapley value function on $\text{FG}_M(N)$.

Proof. The proof proceeds exactly in the same way as that for the Shapley value function [7], and hence omitted.

2. SOLIDARITY VALUE FOR FUZZY COOPERATIVE GAMES

Definition 10. A function $g_i: FG(N) \to (\mathbb{R})^{FG(N)}$ is said to be a solidarity value function on $FG(N)$ if it satisfies the following four axioms.

Axiom SL1.($\boldsymbol{Efficiency}$) If $v \in FG(N)$

$$\sum_{i \in Supp\ U} g_i(U,v) = v(U)$$

Axiom SL2.($\Delta - \boldsymbol{null\ player}$) If $v \in FG(N)$ and $i \in U \in L(N)$ is a $\Delta -$ null player, that is $\Delta^v(T) = 0$ then

$$g_i(U,v) = 0 \quad \forall\ i \in Supp\ U \subset N,$$

Axiom SL3.($\boldsymbol{Symmetry}$) If $v \in FG(N)$ and $i, j \in Supp\ U$ but $i, j \notin Supp\ S$ are symmetric i.e., $v(S \cup U(i)) = v(S \cup U(j))$ holds for any $S \subseteq U$, then

$$g_i(U,v) = g_j(U,v),$$

Axiom SL4.($\boldsymbol{Additivity}$) For $v_1, v_2 \in FG(N)$, define $v_1 + v_2 \in FG(N)$ by $(v_1 + v_2)(U) = v_1(U) + v_2(U)$ for each $S \subseteq U$, then

$$g_i(U, v_1 + v_2) = g_i(U, v_1) + g_i(U, v_2)$$

Theorem 3. *Define a functiong*$: FG(N) \to (\mathbb{R})^{FG(N)}$ *by*

$$g_i(U,v) = \sum_{i \in T_0 \subseteq Supp\ (U)} \frac{(|Supp\ U| - |T_0|)!\,(|T_0| - 1)!}{(|Supp\ U|)!} \Delta^v(T_0), \quad \forall\ i \in Supp\ U$$

Where $\Delta^v(T_0) = \frac{1}{|T_0|} \sum_{k \in T_0} [v(T_0) - v(T_0 \backslash \{k\})]$ *Then the function g is the unique solidarity value function on* $FG(N)$.

Proof. Since $g(U,v)$ depends on U only through $Supp\ U \subseteq N$, it coincides with the classical Nowak-Radzik solidarity value [2] applied to the restricted game on $Supp\ U$. Hence the existence and uniqueness of g remain valid.

Theorem 4. *Define a value function* $\psi^{sol}: \text{FG}_M(N) \to (\mathbb{R})^{\text{FG}_M(N)}$ *satisfying the efficiency, symmetry,* $\Delta - null\ player$ *and additivity axioms, then for* $i \in Supp\ (U)$, *non-empty coalition* $T_0 \subseteq Supp\ (U)$, *and any constant* c,

$$\psi^{sol}{}_i(U, cw_{T_0}) = \begin{cases} c \binom{|S_0|}{|T_0|}^{-1} / |T_0|, & if\ i \in T_0 \\ 0, & otherwise \end{cases}$$

Where $\quad w_{T_0}(S_0) = \begin{cases} \binom{|S_0|}{|T_0|}^{-1}, & if\ S_0 \supset T_0 \\ 0, & otherwise \end{cases}$

Proof. The proof proceeds exactly in the same way as that for the solidarity value function [2], hence omitted.

Theorem 5. *Define a function* $\psi^{sol}: FG_M(N) \to (\mathbb{R})^{FG_M(N)}$ *by*

$\psi^{sol}{}_i(U,v) = \sum_{i\in T_0 \subseteq Supp\ U} \frac{(|T_0|-1)!(|Supp\ U|-|T_0|)!}{|Supp\ U|!}\ \Delta^v(T_0) \quad \forall\ i \in Supp\ U.$

Where

$$\Delta^v(T_0) = \frac{1}{|T_0|}\sum_{k\in T_0}[\left\{\sum_{S_0\subseteq T_0}\prod_{j\in S_0} U(j) \prod_{j\in Supp\ U\setminus S_0}\left(1-U(j)\right)\right\} v(S_0) -$$

$$\left\{\sum_{S_0\subseteq T_0\setminus\{k\}}\prod_{j\in S_0} U(j) \prod_{j\in Supp\ U\setminus S_0}\left(1-U(j)\right)\right\} v(S_0)]$$

Then the function ψ^{sol} *is the unique soli value function on* $\mathrm{FG}_M(N)$.

Proof. **Existence.** From Theorem 3.1 of [7], one can easily obtain existence.

Uniqueness. Let $v \in FG_M(N)$. Since the set $\{w_{T_0} \mid non-empty\ coalition\ T_0 \subseteq Supp\ (U)\}$ is a linear basis of $FG_M(N)$. So v can be expressed by $v = \sum_{T_0\subseteq Supp\ U} c_{T_0} w_{T_0}$, $c_{T_0} \in \mathbb{R}$. By **Axiom SL**4(Additivity), we obtain $\psi^{sol}(U,v) = \sum_{T_0\subseteq Supp\ U} c_{T_0}\ \psi^{sol}(U, w_{T_0})$. Hence ψ^{sol} is uniquely determined on $FG_M(N)$.

Example 1. Let $N = \{1,2,3\}, U = (0.5,\ 0.6,\ 0.8)$ *thus* $Supp\ U = \{1,2,3\}$ and consider the characteristic function v where $v(i) = 0$, $\forall\ i \in \{1,2\}, v(3) = 1, v(1,2) = 3.5, v(2,3) = 0,$ *and* $v(1,3) = 0, v(N) = 5.$ Using multilinear extension formula $v(U) = 1.57$. $\psi^{sol}(U,v) = (0.521,\ 0.521,\ 0.528)$, $\sum_{i\in Supp\ U} \psi^{sol}{}_i(U,v) = v(U) = 1.57$.

3. Conclusion

In this work, we introduced the solidarity value for fuzzy cooperative games using the multilinear extension framework. We established the main axiomatic properties of the proposed value and proved its uniqueness. An example was also provided to illustrate the computation of the solidarity value for a fuzzy coalition. The present work can be further extended by studying solidarity value for the other classes of cooperative games.

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