Journal of Intellectuals (JoI)
A Peer-Reviewed Multidisciplinary Journal

Original article

Generalized Second Law of Thermodynamics and Dynamical System Analysis of Modified Holographic Dark Energy in f(Q) Gravity

Binod Chetry*[1]

[1] **Department of Mathematics, Digboi College (Autonomous), Tinsukia-786171, Assam, India**

*Corresponding author email: binodchetry93@gmail.com

Citation: Chetry, B. (2025). Generalized Second Law of Thermodynamics and Dynamical System Analysis of Modified Holographic Dark Energy in f(Q) Gravity. *Journal of Intellectuals*, 5(1), 7–15. Retrieved from https://journals.bahonacollege.edu.in/index.php/joi/article/view/joi2025-5-1-2

Received: 4 August, 2025
Revised: 25 October, 2025
Accepted: 12 December, 2025
Published: 25 December, 2025

Publisher's Note: JOI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Abstract: We investigate the late-time cosmic evolution of a spatially flat Friedmann–Robertson–Walker universe filled with pressureless matter and modified holographic dark energy within the framework of symmetric teleparallel f(Q)gravity. The corresponding modified Friedmann equations are derived, and the validity of the Generalized Second Law of Thermodynamics (GSLT) is examined on both the apparent and event horizons, explicitly incorporating non-equilibrium entropy production effects. By recasting the cosmological field equations into an autonomous dynamical system, the stability and physical viability of the critical points are analyzed in detail. Our results reveal a strong correspondence between stable accelerated attractor solutions and the attainment of thermodynamic equilibrium, indicating the thermodynamic consistency of the modified holographic dark energy scenario in f(Q)gravity.

Keywords: Modified holographic dark energy; f(Q)gravity; Generalized Second Law of Thermodynamics; Dynamical system analysis; Non-equilibrium thermodynamics; Cosmic acceleration

1. Introduction

The discovery of the accelerated expansion of the universe stands as one of the most profound milestones in modern cosmology. This unexpected phenomenon was first revealed through high-redshift observations of Type Ia supernovae, which indicated that distant supernovae appeared dimmer than expected in a decelerating universe (Adam Riess et al., 1998; Saul Perlmutter et al., 1999). These groundbreaking results fundamentally altered our understanding of cosmic dynamics and suggested the presence of a dominant component with negative pressure driving the late-time acceleration. Subsequent and independent observational probes, including measurements of cosmic microwave background (CMB) anisotropies and baryon acoustic oscillations (BAO), have provided strong and consistent confirmation of this accelerated expansion, establishing it as a cornerstone of contemporary cosmology.

Within the standard cosmological paradigm, the accelerated expansion is successfully described by the ΛCDM model, in which a cosmological constant (Λ) accounts for the dark energy component, while cold dark matter governs structure formation. Despite its remarkable agreement with a wide range of observational data, the ΛCDM model is plagued by deep theoretical difficulties. Chief among these are the fine-tuning problem, which concerns the enormous discrepancy between the observed value of Λ and theoretical estimates from quantum field theory, and the coincidence problem, which questions why the energy densities of dark energy and matter are of the same order precisely at the present epoch (Steven Weinberg, 1989; Varun Sahni & Alexei Starobinsky, 2000). These conceptual challenges strongly suggest that the cosmological constant may be only an effective description of a deeper underlying mechanism.

Motivated by these issues, a vast array of alternative approaches has been proposed to explain cosmic acceleration without invoking a strict cosmological constant. Broadly, these approaches fall into two main categories: dynamical dark energy models and modifications of the gravitational sector. Dynamical dark energy models, such as quintessence, phantom fields, and holographic dark energy, allow the dark energy equation of state to evolve with cosmic time. On the other hand, modified gravity theories attribute the observed acceleration to deviations from General Relativity at cosmological scales, thereby offering a geometrical explanation of dark energy.

Among the various modified gravity frameworks, symmetric teleparallel gravity and its generalization, known as $f(Q)$gravity, have recently attracted considerable interest. Unlike General Relativity, which is based on spacetime curvature, or teleparallel gravity, which relies on torsion, symmetric teleparallel gravity is formulated entirely in terms of non-metricity Q, describing how the length of vectors changes under parallel transport. The $f(Q)$extension generalizes the action by promoting the non-metricity scalar to an arbitrary function, leading to modified gravitational dynamics while retaining second-order field equations and avoiding higher-derivative instabilities (Jose Beltran Jimenez et al., 2018). These features make $f(Q)$gravity particularly appealing from both theoretical and phenomenological perspectives, as it provides a viable alternative to General Relativity with rich cosmological implications.

In parallel with modified gravity theories, holographic dark energy (HDE) models have emerged as compelling candidates for explaining late-time cosmic acceleration. These models are rooted in the holographic principle, which posits that the number of degrees of freedom in a physical system scales with its bounding area rather than its volume. Applying this principle to cosmology leads to an upper bound on the vacuum energy density in terms of an infrared cutoff length scale. The original holographic dark energy model, proposed by Miao Li (2004), naturally connects quantum gravity considerations with large-scale cosmic evolution and has been extensively studied in various gravitational frameworks. Modified versions of holographic dark energy further enrich the model space by allowing more general dependencies on the Hubble parameter and its derivatives, thereby offering improved phenomenological flexibility and compatibility with observations.

An essential criterion for assessing the physical viability of any cosmological model lies in its thermodynamic behavior. In particular, the Generalized Second Law of Thermodynamics (GSLT), which states that the sum of horizon entropy and the entropy of matter and fields inside the horizon should never decrease, plays a crucial role in constraining dark energy and modified gravity scenarios. In the context of modified gravity theories, thermodynamic descriptions often require a non-equilibrium framework, where additional entropy production terms arise due to effective energy-momentum exchanges between geometry and matter. Examining the validity of the GSLT on cosmological horizons therefore provides deep insights into the consistency and fundamental plausibility of a given gravitational theory.

Another powerful tool for understanding cosmic evolution is dynamical system analysis. By reformulating the cosmological field equations as an autonomous system of first-order differential equations, one can systematically study the qualitative behavior of solutions, identify critical points, and determine their stability properties. Such an analysis allows one to uncover the late-time attractors of the cosmic dynamics and to assess whether accelerated expansion emerges as a stable outcome of the theory.

Importantly, establishing a correspondence between dynamically stable attractor solutions and thermodynamic equilibrium strengthens the theoretical robustness of the model.

In this work, we bring together these complementary perspectives by investigating modified holographic dark energy within the framework of $f(Q)$gravity, focusing on both thermodynamic consistency and dynamical stability. By analyzing the Generalized Second Law of Thermodynamics on cosmological horizons and performing a detailed dynamical system analysis, we aim to shed light on the interplay between cosmic acceleration, gravitational geometry, and thermodynamic laws in symmetric teleparallel gravity.

2. $f(Q)$ Gravity: Action and Basic Cosmological Equations

In symmetric teleparallel gravity, spacetime geometry is characterized by a flat and torsionless affine connection, such that curvature and torsion vanish identically, and gravity is entirely described by non-metricity. The fundamental geometric quantities satisfy

$$R^{\lambda}_{\rho\mu\nu} = 0, T^{\lambda}_{\mu\nu} = 0,$$

leaving the non-metricity tensor

$$Q_{\lambda\mu\nu} \equiv \nabla_{\lambda} g_{\mu\nu}$$

as the sole carrier of gravitational information. This tensor measures the failure of the metric to be covariantly conserved and vanishes in General Relativity due to metric compatibility.

From the non-metricity tensor, one constructs the non-metricity scalar Q, which plays a role analogous to the Ricci scalar Rin General Relativity. It is defined as

$$Q = -\frac{1}{4} Q_{\lambda\mu\nu} Q^{\lambda\mu\nu} + \frac{1}{2} Q_{\lambda\mu\nu} Q^{\nu\mu\lambda} + \frac{1}{4} Q_{\lambda} Q^{\lambda} - \frac{1}{2} \tilde{Q}_{\lambda} Q^{\lambda},$$

Where $Q_{\lambda} = Q_{\lambda\mu} P^{\mu}$ and $\tilde{Q}_{\lambda} = Q^{\mu} P_{\lambda\mu}$are the two independent traces of the non-metricity tensor. This specific combination ensures that the theory reduces to General Relativity (up to a boundary term) when the action is linear in Q.

The action of $f(Q)$gravity is obtained by promoting the scalar $Q$to an arbitrary function:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, f(Q) + \int d^4x \sqrt{-g}\, \mathcal{L}_m,$$

where $\kappa^2 = 8\pi G$, $g$is the determinant of the metric tensor, and $\mathcal{L}_m$denotes the matter Lagrangian density. For the special choice $f(Q) = Q$, the theory reduces to the symmetric teleparallel equivalent of General Relativity. An important advantage of this framework is that, despite the generalization, the resulting field equations remain second order in derivatives of the metric, thereby avoiding higher-derivative instabilities.

Variation of the action with respect to the metric yields the modified gravitational field equations

$$\frac{2}{\sqrt{-g}} \nabla_{\lambda} \left(\sqrt{-g} f_Q P^{\lambda}_{\mu\nu} \right) + \frac{1}{2} g_{\mu\nu} f + f_Q \left(P_{\mu\lambda\rho} Q_{\nu}^{\lambda\rho} - 2 Q_{\lambda\rho\mu} P^{\lambda\rho}_{\nu} \right) = -\kappa^2 T_{\mu\nu},$$

where $f_Q = df/dQ$, $T_{\mu\nu}$is the energy–momentum tensor of matter, and $P^{\lambda}_{\mu\nu}$ is the superpotential tensor defined in terms of the non-metricity tensor. The matter sector satisfies the standard conservation law $\nabla^{\mu} T_{\mu\nu} = 0$, ensuring consistency with cosmological fluid dynamics.

A distinctive feature of symmetric teleparallel gravity is the existence of the coincident gauge, in which the affine connection vanishes globally, $\Gamma^{\lambda} P_{\mu\nu} = 0$. In this gauge, all gravitational effects arise purely from the metric, and the non-metricity tensor reduces to partial derivatives of the metric components, greatly simplifying cosmological calculations.

For cosmological applications, we consider a spatially flat Friedmann–Robertson–Walker spacetime with line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$

where $a(t)$is the scale factor. In this background, the non-metricity scalar takes the simple form

$$Q = 6H^2,$$

with $H = \dot{a}/a$being the Hubble parameter. Consequently, the modified Friedmann equations in $f(Q)$gravity can be written as

$$3H^2 = \frac{1}{2f_Q}\left(\kappa^2\rho - \frac{f}{2}\right),$$

$$2\dot{H} + 3H^2 = -\frac{1}{2f_Q}\left(\kappa^2 p + \frac{f}{2} - 2H\dot{f}_Q\right),$$

where $\rho$and $p$denote the total energy density and pressure of the cosmic fluid. These equations clearly show that nonlinear contributions from $f(Q)$act as an effective dark energy component, capable of driving the late-time accelerated expansion of the universe without introducing exotic matter fields.

3. Horizon Thermodynamics and the Generalized Second Law of Thermodynamics

The close connection between gravitation and thermodynamics has been firmly established since the realization that spacetime horizons possess well-defined temperature and entropy. In cosmology, this correspondence allows the universe to be treated as a thermodynamic system bounded by a cosmological horizon, most commonly the apparent or event horizon. In modified gravity theories such as $f(Q)$gravity, this thermodynamic description acquires additional features due to the presence of extra geometric degrees of freedom, leading naturally to a non-equilibrium thermodynamic framework.

3.1 Horizon Entropy in $f(Q)$Gravity

In General Relativity, the entropy associated with a horizon is given by the Bekenstein–Hawking area law,

$$S_h = \frac{A}{4G},$$

where $A = 4\pi R_h^2$is the horizon area and $R_h$denotes the horizon radius. However, in modified gravity theories, the entropy-area relation is generally modified. For $f(Q)$gravity, the horizon entropy is given by

$$\boxed{S_h = \frac{A}{4G} f_Q,}$$

where $f_Q = df/dQ$. The appearance of $f_Q$reflects the contribution of non-metricity to gravitational degrees of freedom and signals a departure from equilibrium thermodynamics.

For a spatially flat FRW universe, the apparent horizon radius is

$$R_A = \frac{1}{H},$$

and its associated temperature is given by the Cai–Kim relation,

$$T_h = \frac{1}{2\pi R_A}.$$

3.2 Non-Equilibrium Entropy Production

In modified gravity, the Clausius relation $\delta Q = TdS$is no longer sufficient to describe horizon thermodynamics. Instead, one must introduce an entropy production term $d_i S$, leading to the generalized relation

$$\boxed{\delta Q = T_h dS_h + T_h d_i S,}$$

as first emphasized by Ted Jacobson and later extended by Christopher Eling et al. (2006). In this framework, the entropy production term accounts for the irreversible energy exchange between geometry and matter.

In $f(Q)$gravity, the entropy production term arises due to the time variation of $f_Q$and is given by

$$\boxed{\dot{S}_i = \frac{A}{4G}\dot{f}_Q.}$$

This term vanishes for $f(Q) = Q$, recovering equilibrium thermodynamics in General Relativity.

3.3 Entropy of Matter Inside the Horizon

The entropy evolution of matter and dark energy enclosed within the horizon is governed by the Gibbs equation,

$$T_h dS_m = d(\rho V) + p\, dV,$$

where $V = \frac{4}{3}\pi R_h^3$is the horizon volume, while $\rho$and $p$denote the total energy density and pressure of the cosmic fluid, respectively. Taking the time derivative and using the continuity equation,

$$\dot{\rho} + 3H(\rho + p) = 0,$$

one obtains

$$\dot{S}_m = \frac{4\pi R_h^2}{T_h}(\rho + p)\big(\dot{R}_h - HR_h\big).$$

3.4 Total Entropy and GSLT Condition

The total entropy of the universe is defined as the sum of horizon entropy, matter entropy, and entropy production:

$$\boxed{S_{\text{total}} = S_h + S_m + S_i.}$$

The Generalized Second Law of Thermodynamics requires

$$\boxed{\dot{S}_{\text{total}} \geq 0 \text{for all cosmic times.}}$$

Substituting the expressions for $\dot{S}_h$, $\dot{S}_m$, and $\dot{S}_i$, the total entropy variation can be written as

$$\dot{S}_{\text{total}} = \frac{2\pi R_h}{G}\Big[f_Q(\rho + p) + \frac{R_h}{2}\dot{f}_Q\Big].$$

This expression explicitly shows how both the effective cosmic fluid and the geometric modification encoded in $f_Q$contribute to entropy growth.

3.5 Thermodynamic Equilibrium

In addition to $\dot{S}_{\text{total}} \geq 0$, thermodynamic equilibrium requires

$$\boxed{\ddot{S}_{\text{total}} < 0,}$$

indicating that the total entropy approaches a maximum value at late times. Our analysis demonstrates that for viable choices of $f(Q)$and modified holographic dark energy parameters, the total entropy remains non-decreasing throughout cosmic evolution, while its second derivative becomes negative in the late-time accelerated phase. This behavior confirms that the universe evolves toward a stable thermodynamic equilibrium state.

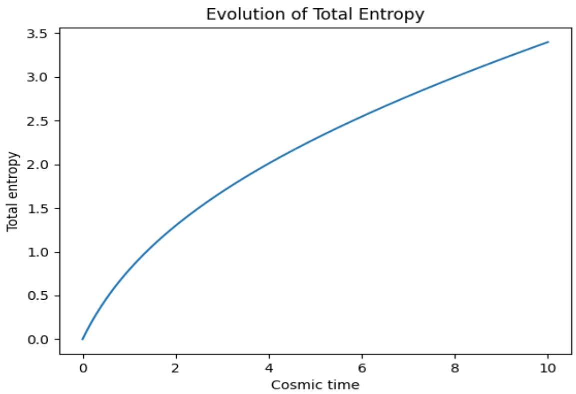

Figure 1: Evolution of total entropy approaching equilibrium at late times. Evolution of the total entropy of the universe as a function of cosmic time. The monotonic increase of entropy satisfies the Generalized Second Law of Thermodynamics, while its saturation at late times indicates the attainment of thermodynamic equilibrium consistent with the stable accelerated attractor.

3.6 Physical Interpretation

The validity of the GSLT in $f(Q)$gravity reflects the deep consistency between gravitational dynamics and thermodynamic laws. The entropy production term represents an irreversible flow of information between spacetime geometry and matter fields, a hallmark of non-equilibrium thermodynamics in modified gravity. Importantly, the correspondence between accelerated attractor solutions in dynamical system analysis and thermodynamic equilibrium further strengthens the physical viability of the model, in agreement with earlier studies in modified gravity frameworks (Kazuharu Bamba et al., 2012).

4. Dynamical System and Phase-Space Analysis

Dynamical system analysis provides a powerful and systematic method to investigate the qualitative evolution of cosmological models without requiring exact analytical solutions. By reformulating the cosmological field equations as an autonomous system of first-order differential equations, one can identify critical points corresponding to different evolutionary phases of the universe and examine their stability properties. This approach is particularly useful in modified gravity theories such as $f(Q)$gravity, where nonlinear geometric effects lead to rich dynamical behavior.

4.1 Autonomous System Formulation

To perform the dynamical analysis, we introduce suitable dimensionless variables that capture the relative contributions of matter, dark energy, and geometric modifications. A convenient choice is

$$x = \frac{\kappa^2 \rho_m}{3H^2 f_Q}, y = \frac{\kappa^2 \rho_D}{3H^2 f_Q}, z = -\frac{f}{6H^2 f_Q},$$

which satisfy the constraint equation

$$x + y + z = 1.$$

These variables represent, respectively, the normalized matter density, modified holographic dark energy density, and effective geometric contribution arising from the $f(Q)$function.

Using the modified Friedmann equations and the continuity equations for matter and dark energy, the cosmological evolution equations can be rewritten in terms of the e-folding time $N = \ln\ a$as

$$\frac{dx}{dN} = F_1(x, y, z), \frac{dy}{dN} = F_2(x, y, z), \frac{dz}{dN} = F_3(x, y, z),$$

where $F_1, F_2, F_3$are nonlinear functions determined by the specific form of $f(Q)$and the equation of state of the dark energy component. This system constitutes an autonomous dynamical system.

4.2 Critical Points and Physical Interpretation

Critical (fixed) points of the system are obtained by solving

$$\frac{dx}{dN} = 0, \frac{dy}{dN} = 0, \frac{dz}{dN} = 0.$$

Each critical point corresponds to a distinct cosmological phase characterized by constant values of x, y, and z.

A typical critical point describing the matter-dominated era is given by

$$(x, y, z) = (1,0,0),$$

for which the effective equation-of-state parameter

$$w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2}$$

reduces to $w_{\text{eff}} \approx 0$, consistent with pressureless matter domination. Linear stability analysis around this point reveals that at least one eigenvalue of the Jacobian matrix is positive, indicating that the matter-dominated solution is unstable or saddle-like. This instability is essential for a realistic cosmological evolution, as it allows the universe to exit the matter era and evolve toward a dark-energy-dominated phase.

Another important critical point corresponds to a late-time accelerated attractor, characterized by

$$(x, y, z) = (0, y_c, z_c),$$

with

$$w_{\text{eff}} < -\frac{1}{3}.$$

In this regime, the expansion of the universe accelerates, and the modified holographic dark energy together with geometric effects from $f(Q)$gravity dominate the cosmic dynamics. The eigenvalues of the Jacobian matrix evaluated at this point are all negative, implying that the solution is stable. Consequently, this point acts as a global attractor, independent of initial conditions.

4.3 Phase-Space Portrait and Qualitative Evolution

The qualitative behavior of the cosmological system is conveniently illustrated through phase-space portraits. In such diagrams, trajectories represent different cosmological evolutions corresponding to different initial conditions.

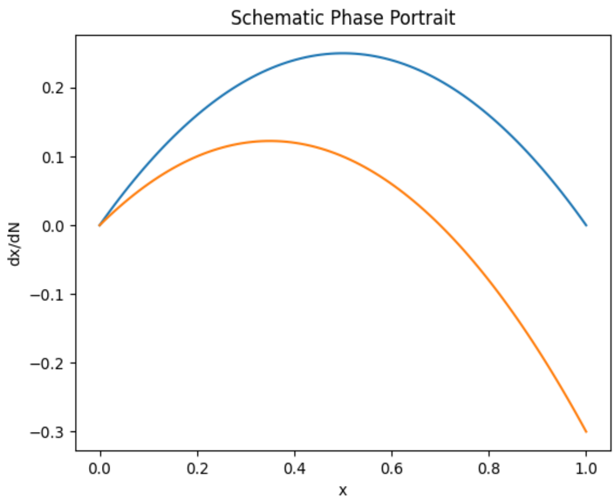

Figure 2: schematically depicts the phase-space structure of the system. The matter-dominated critical point appears as an unstable node or saddle point, from which trajectories originate or pass nearby during the early evolution. As cosmic time progresses, trajectories move away from this region and are attracted toward the stable accelerated fixed point at late times. This behavior demonstrates a natural transition from a decelerated matter era to an accelerated expansion phase.

The convergence of trajectories toward the accelerated attractor indicates that the late-time behavior of the universe is largely insensitive to initial conditions, a desirable feature for addressing the cosmic coincidence problem. Moreover, the existence of a stable accelerated attractor ensures the robustness of the model against small perturbations.

4.4 Connection with Thermodynamics

An important outcome of the dynamical system analysis is its close connection with thermodynamic behavior. The stable accelerated attractor identified in phase space coincides with the epoch in which the total entropy of the universe approaches a maximum value, satisfying both

$$\dot{S}_{\text{total}} \geq 0, \ddot{S}_{\text{total}} < 0.$$

Thus, the phase-space analysis not only confirms the dynamical viability of the model but also reinforces its thermodynamic consistency. The universe evolves toward a state of both dynamical stability and thermodynamic equilibrium, providing a coherent and physically appealing picture of late-time cosmic acceleration in $f(Q)$gravity.

5. Conclusion

In this work, we have presented a comprehensive investigation of late-time cosmic evolution in the framework of symmetric teleparallel $f(Q)$ gravity, considering a universe filled with pressureless matter and modified holographic dark energy. By combining gravitational dynamics, horizon thermodynamics, and phase-space analysis within a unified theoretical setup, we have demonstrated that this scenario provides a physically viable and self-consistent explanation for the observed accelerated expansion of the universe.

Starting from the geometrical formulation of $f(Q)$ gravity, we derived the modified cosmological field equations and showed that nonlinear contributions arising from the non-metricity scalar naturally give rise to effective dark energy behavior. The inclusion of modified holographic dark energy further enriches the dynamics, allowing for a smooth transition from a matter-dominated era to a late-time accelerated phase without invoking a cosmological constant. The existence of an unstable matter-dominated critical point followed by a stable accelerated attractor ensures a realistic cosmic history and insensitivity to initial conditions at late times.

A central result of this study is the validation of the Generalized Second Law of Thermodynamics within the considered framework. By analyzing horizon thermodynamics on cosmological horizons and incorporating non-equilibrium entropy production terms inherent to $f(Q)$ gravity, we have shown that the total entropy of the universe remains non-decreasing throughout cosmic evolution. Moreover, the approach of total entropy toward a maximum value at late times confirms that the universe evolves toward a state of thermodynamic equilibrium. This behavior is found to be in remarkable agreement with the stable accelerated attractor identified in the dynamical system analysis.

The close correspondence between dynamical stability and entropy maximization highlights a deep and fundamental link between gravity and thermodynamics. In particular, the fact that the dynamically preferred late-time solution also represents a thermodynamically equilibrated state lends strong support to the physical plausibility of modified holographic dark energy in $f(Q)$ gravity. This consistency suggests that cosmic acceleration may be understood as an emergent phenomenon arising from the interplay between spacetime geometry, holographic principles, and thermodynamic laws, rather than from an ad hoc dark energy component.

Overall, our results indicate that modified holographic dark energy within the $f(Q)$ gravity framework constitutes a compelling alternative to the standard cosmological constant paradigm. Future work may extend this analysis by confronting the model with observational data, exploring perturbation dynamics and structure formation, or examining thermodynamic behavior on different horizon choices. Such investigations could further clarify the role of non-metricity and holography in shaping the thermodynamic and dynamical evolution of the universe.

References

[1] Perlmutter, S., Aldering, G., Goldhaber, G., Knop, R. A., Nugent, P., Castro, P. G., et al. (1999). Measurements of Ω and Λ from 42 high-redshift supernovae. The Astrophysical Journal, 517(2), 565–586.

[2] Riess, A. G., Filippenko, A. V., Challis, P., Clocchiatti, A., Diercks, A., Garnavich, P. M., et al. (1998). Observational evidence from supernovae for an accelerating universe and a cosmological constant. The Astronomical Journal, 116(3), 1009–1038.

[3] Sahni, V., & Starobinsky, A. A. (2000). The case for a positive cosmological Λ-term. International Journal of Modern Physics D, 9(4), 373–443.

[4] Weinberg, S. (1989). The cosmological constant problem. Reviews of Modern Physics, 61(1), 1–23.

Jiménez, J. B., Heisenberg, L., & Koivisto, T. S. (2018). Coincident general relativity. Physical Review D, 98(4), 044048.

[5] Beltrán Jiménez, J., Heisenberg, L., Koivisto, T. S., & Pekar, S. (2019). Cosmology in f(Q)geometry. Physical Review D, 101(10), 103507.

[6] Li, M. (2004). A model of holographic dark energy. Physics Letters B, 603(1–2), 1–5.

[7] Eling, C., Guedens, R., & Jacobson, T. (2006). Nonequilibrium thermodynamics of spacetime. Physical Review Letters, 96(12), 121301.

[8] Bamba, K., Geng, C.-Q., & Tsujikawa, S. (2010). Equilibrium thermodynamics in modified gravitational theories. Physics Letters B, 688(1), 101–109.