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A Comprehensive Study of the Kantowski-Sachs Cosmological Model and Its Thermodynamic Analysis

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Abstract: This paper deals with the study of Kantowski-Sachs cosmological model with New Holographic Dark Energy (NHDE) in the framework of General Relativity (GR). To find the system of field equations entirely, simple parametrization of average scale factor $a(t) = exp\{(\gamma t + \delta)^l\}$ as suggested by Mishra and Dua has been adopted. Some parameters such as matter energy density, NHDE density, Hubble parameter, etc. has been discussed physically and graphically. In addition, the Generalized Second Law (GSL) of thermodynamics is validated. The bulk viscosity's role in maintaining thermodynamic consistency is also examined. The results indicate that the Kantowski-Sachs model is consistent with basic thermodynamic laws and gives insights into how cosmic anisotropy and thermodynamics interact. Quantum corrections and their effect on the thermodynamic evolution of the universe may be explored in future research. The outcomes obtained were found to agree with the present-day observations.

Keywords: Kantowski-Sachs, NHDE, Thermodynamics, GSL, GR

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1. Introduction

Observational evidence from all observations [1-6] is pointing out that our Universe is experiencing an accelerating expansion. All of these observations implies that our Universe consists of an unseen form of energy called as dark energy (DE). DE, a kind of exotic energy with negative pressure makes up roughly 70% of the Universe's total energy density [7]. The most straightforward candidate for DE is the cosmological constant Λ with EOS parameter $\omega = \frac{p}{\rho} = -1$ where p is pressure and ϱ is energy density of DE. But it has the fine-tuning problem and cosmic coincidence problem [8, 9]. There are a number of candidates to take the role of DE which is the dominant component of the Universe. Some of these include quintessence [10], k-essence [11], holographic dark energy [12], phantom [13, 14], tachyon [15], etc.

Based on holographic principle, holographic dark energy (HDE) was initially proposed by [16, 17]. From this principle, the entropy of the system varies not with volume, but with surface area $L^2$and came to a conclusion that in quantum field theory a cut-off at a short distance corresponds to a cut-off at long distance because of the constraint provided by the black hole formation [18]. Taking ρ_{HDE} as quantum zero-point energy density due to a cut-off at short distances in a region of size L, total energy density would not be greater than the black hole mass of the same volume, yielding $L^3\rho_{HDE} \ \leq \ LM_p^2$. The maximum value L permitted is the saturating value of this inequality, with the HDE density being $\rho_{HDE} \ = \ 3c^2M_p^2L^{-2}$, M_p being the reduced Planck mass with $M_p^2 \ = \ 8\pi G$ and $3c^2$ being the numerical factor. Gao et al. [19] derived a holographic dark energy model. In this model the future event horizon is replaced by the inverse of the Ricci scalar curvature. This model is 'Ricci Dark Energy' (RDE) model. Granda and Oliveros [20, 21] presented a new holographic Ricci dark energy with dark energy density as $\rho_N \ = \ 3M_p^2(\alpha H^2 + \beta\dot{H})$ where α and β are constants and dot (.) represents derivative w.r.t. cosmic time t. Various authors have examined different types of cosmological models in various theories of gravity [22–25].

The Kantowski-Sachs cosmological model is very important in the investi¬gation of anisotropic universe models, and especially in situations such as early universe evolution, quantum cosmology, and theories of modified gravity. One aspect of the model that is important is its thermodynamic behaviour, especially the evolution of entropy, the first law of thermodynamics, and the Generalized Second Law of Thermodynamics (GSL) under bulk viscosity and quantum corrections. The relation between general relativity and thermodynamics was first established by Bekenstein in 1973 [26] and further developed by Hawking in 1975 [27], who introduced the entropy-area relation for black hole horizons:

$$S \ = \ \frac{\kappa_B A}{4G}$$

where κ_B is the Boltzman constant, A is the horizon area. This concept was extended to cosmological horizons [28, 29] to determine that the universe's horizon may be considered as a thermodynamic system. The thermodynamic characteristics of expanding universes have been studied in detail in isotropic contexts, like the FLRW model [26, 27]. But in anisotropic cosmologies such as the Kantowski-Sachs model, entropy evolution and the validity of thermodynamic laws need to be explored further. A number of papers have extended the usual thermodynamic analysis incorporating bulk viscosity, apparent horizon entropy, and energy conditions [29, 30].

In this paper, we have explored Kantowski-Sachs cosmological model with NHDE in GR. Section 2 deals with the metric and the field equations. Field equations solutions were derived in Section 3. In section 4, we present thermo¬dynamic analysis in Kantowski-Sachs Model. At last, the paper concludes with concluding remarks in Section 5.

2. Metric and Field Equations:

Kantowski-Sachs metric is given by

$$ds^2 = dt^2 - A^2dr^2 - B^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (1)$$

where cosmic scale factors A and B are functions of cosmic time t.

Einstein's field equations are given by

$$R_{ij} - \frac{1}{2}g_{ij}R = -T_{ij} \qquad (2)$$

where R_{ij} is the Ricci tensor and R is the Ricci scalar.

The energy momentum tensor for dark matter (DM) $\acute{T}_j^i$is given by

$$\acute{T}_j^i = diag[\rho_m, 0,0,0] \qquad (3)$$

where ρ_m is the matter energy density.

The energy momentum tensor for NHDE T¯_j^i is given by

$$\bar{T}_j^i = diag[1, -\omega_N, -\omega_N, -\omega_N]\rho_N \quad (4)$$

where ρ_N is the energy density of NHDE and $\omega_N = \frac{p_N}{\rho_N}$ is the EOS parameter of NHDE.

$$T_j^i = \acute{T}_j^i + \bar{T}_j^i \quad (5)$$

Using Eqs. (3)-(5), Einstein field equations (2) for the metric (1) takes the form

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} = -\omega_N \rho_N \quad (6)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -\omega_N \rho_N \quad (7)$$

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} = \rho_m + \rho_N \quad (8)$$

where overhead dot $(.)$ denotes differentiation w.r.to cosmic time t.

3. Solutions of Field Equations:

The spatial volume V and average scale factor a are defined as

$$V = a^3 = AB^2 \quad (9)$$

The Hubble parameter H is calculated as

$$H = \frac{\dot{a}}{a} = \frac{\dot{V}}{3V} = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) \quad (10)$$

The deceleration parameter q is defined as

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \quad (11)$$

The anisotropy parameter A_p is defined as

$$A_p = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \quad (12)$$

Eqs. (6)-(8) are three field equations with five unknowns A, B, ρ_m, ρ_N and ω_N. So, we need two extra conditions to solve the field equations completely. These are as follows:

i. The new holographic dark energy density (NHDE) with new IR cut-off as suggested by Granda and Oliveros [20] is given by

$$\rho_N = 3M_p^2(\alpha H^2 + \beta \dot{H}) \quad (13)$$

with $M_p^{-2} = 8\pi G = 1$.

Also, α and β are the parameters of the model which need to meet the expectations forced by the current observational information.

ii. Mishra and Dua [31] proposed a simple parametrization of average scale factor $a(t)$ as

$$a = exp\{(\gamma t + \delta)^l\} \tag{14}$$

where $\gamma, \delta > 0$ and $0 < l < 1$ are arbitrary constants.

Solving Eqs. (6) and (7), we get

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{u_0}{V} exp\left(\int \frac{\frac{-1}{B^2}}{\frac{\dot{B}}{B} - \frac{\dot{A}}{A}} dt\right) \tag{15}$$

where u_0 is a constant of integration.

Following Adhav [32], we assume

$$\frac{\dot{B}}{B} - \frac{\dot{A}}{A} = \frac{1}{B^2} \tag{16}$$

Using Eq. (16), Eq. (15) takes the form

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{u_0}{V} e^{-t} \tag{17}$$

Integrating Eq. (17), we obtain

$$A = u_1 B exp\left\{u_0 \int \frac{e^{-t}}{[exp\{(\gamma t + \delta)^l\}]^3} dt\right\} \tag{18}$$

where u_1 is a constant of integration.

Using Eqs. (9) and (14) in Eq. (18), we get the cosmic scale factors A and B as

$$A = exp\{(\gamma t + \delta)^l\} u_1^{2/3} exp\left\{\frac{2u_0}{3} \int \frac{e^{-t}}{[exp\{(\gamma t + \delta)^l\}]^3} dt\right\} \tag{19}$$

$$B = exp\{(\gamma t + \delta)^l\} u_1^{-1/3} exp\left\{\frac{-u_0}{3} \int \frac{e^{-t}}{[exp\{(\gamma t + \delta)^l\}]^3} dt\right\} \tag{20}$$

The Hubble parameter H is calculated as

$$H = l\gamma(\gamma t + \delta)^{l-1} \tag{21}$$

The NHDE density ρ_N is obtained as

$$\rho_N = 3[\alpha l^2 \gamma^2 (\gamma t + \delta)^{2l-2} + \beta l(l-1)\gamma^2(\gamma t + \delta)^{l-2}] \tag{22}$$

The deceleration parameter q is obtained as

$$q = -1 - \left(\frac{l-1}{l}\right)(\gamma t + \delta)^{-l} \tag{23}$$

The Anisotropy parameter A_p is calculated as

$$A_p = \frac{2u_0^2 e^{-2t-6(\gamma t+\delta)^l}}{9l^2\gamma^2(\gamma t+\delta)^{2l-2}} \tag{24}$$

Using Eqs. (19), (20) and (22) in Eq. (6), we get the EOS parameter ω_N as

$$2l(l-1)\gamma^2(\gamma t+\delta)^{l-2} + \frac{2u_0}{3}\left[1+3l\gamma(\gamma t+\delta)^{l-1}\right]\frac{e^{-t}}{\left[exp\{(\gamma t+\delta)^l\}\right]^3} + 3\left\{l\gamma(\gamma t+\delta)^{l-1} - \frac{e^{-t}}{\left[exp\{(\gamma t+\delta)^l\}\right]^3}\right\}^2 + \frac{1}{B^2} = -\omega_N \rho_N \tag{25}$$

Using Eqs. (19), (20) and (22) in Eq. (7), we get the matter energy density ρ_m as

$$2\left\{l\gamma(\gamma t+\delta)^{l-1} + \frac{2u_0 e^{-t}}{3\left[exp\{(\gamma t+\delta)^l\}\right]^3}\right\}\left\{l\gamma(\gamma t+\delta)^{l-1} - \frac{u_0 e^{-t}}{3\left[exp\{(\gamma t+\delta)^l\}\right]^3}\right\} + \left\{l\gamma(\gamma t+\delta)^{l-1} - \frac{u_0 e^{-t}}{3\left[exp\{(\gamma t+\delta)^l\}\right]^3}\right\}^2 + \frac{1}{B^2} - \rho_N = \rho_m \tag{26}$$

4. Thermodynamics in the Kantowski-Sachs Model

Thermodynamic laws play a crucial role in understanding cosmic evolution. The Kantowski-Sachs model, initially proposed by Kantowski & Sachs in 1966 [33], represents a spatially homogeneous but anisotropic cosmology. Early studies of anisotropic thermodynamics include Misner in 1968, who proposed that entropy production due to shear viscosity could explain the isotropization of the universe [34].

Recent works of Setare in 2007 and Mohseni et al. in 2019 have explored the thermodynamic behaviour of anisotropic models in the context of horizon thermodynamics [35]. The apparent horizon in anisotropic spacetimes, unlike the event horizon, dynamically evolves with time, influencing entropy generation.

The first law of thermodynamics,

$$dE = TdS + WdV \tag{27}$$

has been applied to cosmological horizons by several researchers. Akbar & Cai in 2006 [29] derived a connection between the first law and Einsteins field equations, showing that the Friedmann equation can be rewritten in thermodynamic terms. Eling et al. in 2006 [30] further established the connection between the Einstein equation and non-equilibrium thermodynamics. On the other hand, the Generalized Second Law (GSL) states that the sum of the horizon entropy and the entropy of the fluid inside the horizon must always increase:

$$S_{total} = S_h + S_{fluid} \geq 0 \tag{28}$$

The equation ensures that both the horizon entropy and fluid entropy contribute to maintaining thermodynamic consistency. This principle was tested in various cosmological settings, including isotropic [35] and anisotropic universes [36]. It was found that bulk viscosity plays a key role in ensuring the validity of the GSL in anisotropic models.

4.1 Apparent Horizon and Energy Flow

The apparent horizon in the Kantowski-Sachs spacetime is given by:

$$R_A = B(t) \tag{29}$$

The total energy inside the horizon is:

$$E = \rho V, V = \frac{4}{3}\pi B^3 \tag{30}$$

Taking time derivative, we get

$$\dot{E} = 4\pi B^2 \rho \dot{B} + \frac{4}{3}\pi B^3 \dot{\rho} \tag{31}$$

Using the continuity equation:

$$\dot{\rho} + (\rho + p)\theta = 0, \quad \theta = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} \tag{32}$$

We get

$$\dot{E} = 4\pi B^2 \rho \dot{B} - \frac{4}{3}\pi B^3 (\rho + p)\theta \tag{33}$$

Using first law of thermodynamics and solving for entropy evolution, we have

$$TdS = [4\pi B^2 \dot{B}(\rho - \frac{1}{2}(\rho + p)) - \frac{4}{3}\pi B^3 (\rho + p)\theta]dt \tag{35}$$

4.2 Entropy Evolution and the generalized second law (GSL)

The horizon entropy is given by the Bekenstein-Hawking entropy formula:

$$S_h = \frac{\kappa_B A}{4G} = \frac{\kappa_B}{4G}(4\pi B^2) = \frac{\kappa_B \pi B^2}{G} \tag{36}$$

Taking the time derivative, we have

$$\dot{S_h} = \frac{2\kappa_B \pi B \dot{B}}{G} \tag{37}$$

For the total entropy (horizon+fluid), we require:

$$S_{total} = S_h + S_{fluid} \geq 0 \tag{38}$$

where the fluid entropy production is

$$S_{fluid} = \frac{4\pi B^2}{T}(\rho + p)\dot{B} \quad (39)$$

Thus, the GSL condition becomes:

$$\frac{2\kappa_B \pi B \dot{B}}{G} + \frac{4\pi B^2}{T}(\rho + p)\dot{B} \geq 0 \quad (40)$$

This result ensures that entropy production is positive, preserving the second law of thermodynamics.

4.3 Bulk Viscosity and Effective Pressure

If bulk viscosity is included, the effective pressure is:

$$p_{eff} = p - \zeta\theta \quad (41)$$

where p_{eff} is the effective pressure, modified by bulk viscosity and ζ is the bulk viscosity coefficient. Also, $\zeta\theta$ represents viscous resistance to cosmic expansion. The energy conservation equation modifies to:

$$\dot{\rho} + (\rho + p_{eff})\theta = 0 \quad (42)$$

Substituting p_{eff}, we get

$$\dot{\rho} + (\rho + p - \zeta\theta)\theta = 0 \quad (43)$$

The entropy production due to viscosity $TdS_{viscosity}$ ensures that dissipation due to viscosity contributes to the overall entropy increase and is given by

$$TdS_{viscosity} = -V\zeta\theta^2 dt \quad (44)$$

Since $\zeta > 0$, this guarantees that:

$$S_{total} \geq 0. \quad (45)$$

ensuring that the GSL always holds in the presence of bulk viscosity.

5. **Graphical Discussions:**

The numerical values used in the graphs are $\gamma = 1.2, l = 0.4, \delta = 0.5, \alpha = 0.85, \beta = 0.482, u_0 = 0.03$ and $u_1 = 0.02$.

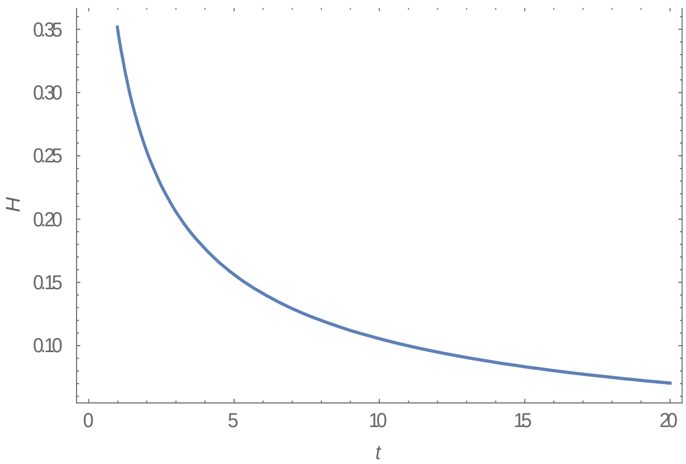

Fig. 1: The plot of Hubble parameter H verses cosmic time t.

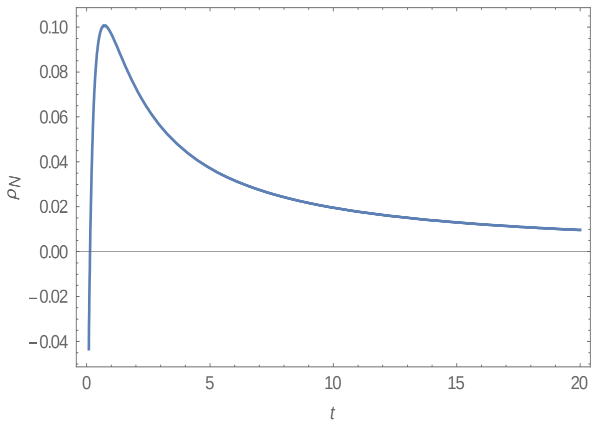

Fig. 2: The variation of ρ_N verses t.

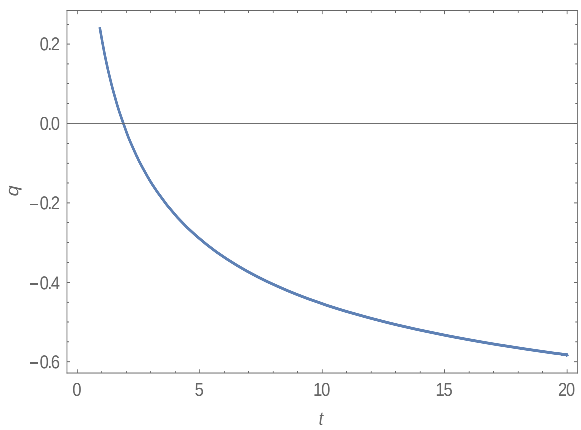

Fig. 3: The evolution of q verses t.

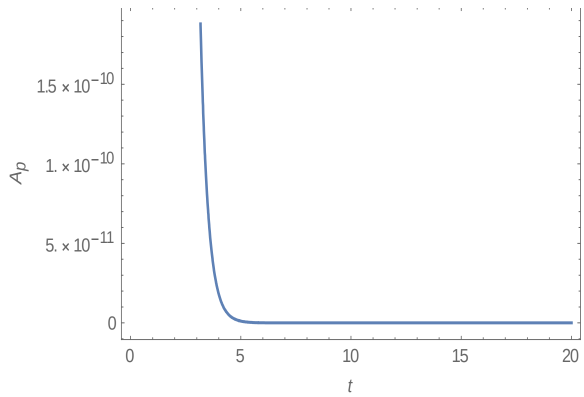

Fig. 4: The variation of A_p verses t.

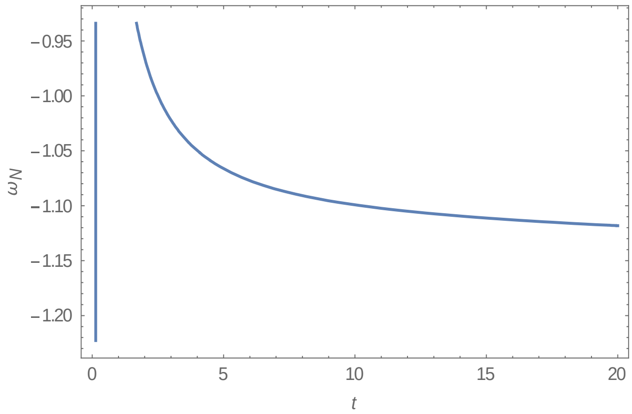

Fig. 5: The graph of ω_N verses cosmic time t.

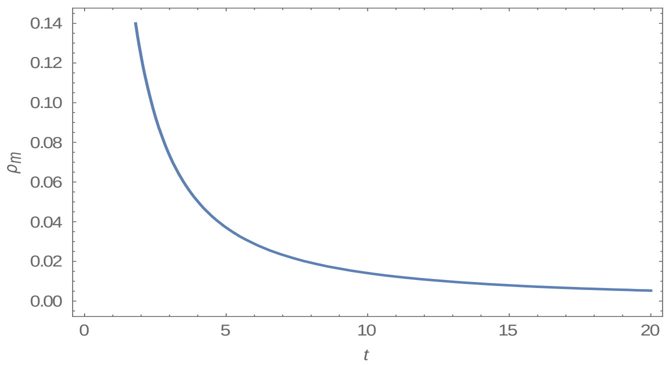

Fig. 6: The plot of ρ_m verses t.

6. Conclusions:

Here we have discussed Kantowski-Sachs cosmological model with NHDE in the framework of GR. The main features of the cosmological model are point-wise listed below:

- The deceleration parameter q yields two phases of the Universe. At the initial stage, the sign of q is positive which indicates the model is decelerating whereas the sign of q is negative which indicates the model is accelerating. Thus, our model shows early cosmic deceleration to present cosmic acceleration of the Universe.
- The Hubble parameter H is a decreasing function of cosmic time t and ultimately approaches to a small value as cosmic time evolves.
- Both NHDE and matter energy densities ρ_N and ρ_m decreases and ultimately approaches to zero at the later stage of the Universe.
- $A_p \to 0$ as cosmic time t evolves. Thus, our Universe shows isotropic behavior throughout the expansion of the Universe.
- The EOS parameter $\omega_N < -1$ at the later age of the Universe which indicates our model behaves like phantom dark energy [13].
- The first law of thermodynamics is consistent with the Kantowski-Sachs model, showing how energy flows across the horizon.
- The GSL holds, provided bulk viscosity is included, which ensures the net entropy increase in an expanding universe.
- Future studies should explore quantum corrections (logarithmic entropy terms) and higher-dimensional models for deeper insights.

The results are worth to pay attention in astrophysical observations.

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